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Dissertation.

Submitted to the Board of University Studies of the Johns Hopkins University in conformity with the requirement for the degree of Doctor of Philosophy.

A Study of the Ternary Quartic in its Relation to Conics.

By Arthur Byron Coble

Mar. 1, 1902.

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A characteristic feature of the plane curve of even order, $2n$, is the one to one correspondence set up by it between the curves of class n & order n. Given a fundamental curve of order $2n$, every curve of class n has a definite polar curve of order n. every curve of order n has "associated" with it a curve of class n whose polar it is.

In the following this correspondence will be studied. On the one hand there will be considered a generalization of the configuration known as the self polar triangle in the case $n=1$. On the other there will be treated some simultaneous invariants & covariants of the system made up of a curve of order $2n$ & a curve of class $2n$, the method used being an extension to the ternary domain of that employed by Hilbert* for binary forms.

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* Math. Ann. Bd 28 p 381

The first section will be devoted to the investigation of an error contained in a statement of Clifford, an error pointed out by Prof. Morley in his lectures.* The facts there obtained will naturally lead us to the more general treatment which follows.

Forms of order n will be denoted in general by f^n, g^n, etc. or symbolically by α_x^n, β_x^n, …; those of class n, by F^n, G^n, etc. or by a_ξ^n, b_ξ^n, … . The notation of Salmon (Higher Plane Curves) for the forms allied to the quartic f^4 will as far as possible be used. C_{ikl} will be the general expression for an invariantive form of f^4 of degree i, order k, and class l.

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* J. H. U., May, 1901.

Section I

A Special Class of Quartic Curves

In the collected works of Clifford the following statement is found:– "The quartic f^4 has a contravariant of degree 3 & class 4 ($\Phi = C_{304}$), the evectant of the invariant B, which is possessed of this property, that every conic, c_ξ^2, has a polar as to f^4 whose polar as to Φ is again c_ξ^2.

We shall prove first

"Theorem I :– For the *general* quartic $f^4 = \alpha_x^4$ there exists no quartic, $F^4 \equiv a_\xi^4$, which has with regard to f^4 the above described property.

For suppose that F^4 has with f^4 the above property & let c_ξ^2 be any line conic. Then must the equation

$$(1) \qquad (c\alpha)^2(\alpha a)^2 a_\xi^2 \equiv c_\xi^2$$

hold identically for all values of the ξ's & c's. In strictness a factor λ should be introduced before c_ξ^2 in (1), a factor however which is necessarily independent of c_ξ^2

and which may be supposed incorporated with the coefficients of f.

Equating coefficients of $\xi_i \xi_k$ in (1) we have six equations of the type

$$(2) \qquad (c\alpha)^2(\alpha a)^2 a_{ik} = c_{ik}$$

which are identities in the quantities c_{ik}.

If in each of the six equations (2) we equate coefficients of c_{11} the following system is obtained:

$$a_{1111}\alpha_{1111} + a_{1122}\alpha_{1122} + a_{1133}\alpha_{1133} + 2a_{1123}\alpha_{1123} + 2a_{1131}\alpha_{1131} + 2a_{1112}\alpha_{1112} = 1$$

$$a_{1111}\alpha_{2211} + a_{1122}\alpha_{2222} + a_{1133}\alpha_{2233} + 2a_{1123}\alpha_{2223} + 2a_{1131}\alpha_{2231} + 2a_{1112}\alpha_{2212} = 0$$

(3) -

$$a_{1111}\alpha_{1211} + a_{1122}\alpha_{1222} + a_{1133}\alpha_{1233} + 2a_{1123}\alpha_{1223} + 2a_{1131}\alpha_{1231} + 2a_{1112}\alpha_{1212} = 0$$

The determinant of this system in the six unknowns $a_{1111}, a_{1122}, a_{1133}, 2a_{1123}, 2a_{1131}, 2a_{1112}$ is the invariant B which for the general quantic f is not equal to zero. If we denote the minor of α_{iklm} in B by A_{iklm} we have at once from (3)

$$a_{1111} = \frac{A_{1111}}{B}, \quad a_{1122} = \frac{A_{1122}}{B}, \quad a_{1133} = \frac{A_{1133}}{B},$$

$$a_{1123} = \frac{A_{1123}}{2B}, \quad a_{1131} = \frac{A_{1131}}{2B}, \quad a_{1112} = \frac{A_{1112}}{2B}$$

Similarly, by equating coefficients of C_{22} in (2) we have that

$$a_{2211} = \frac{A_{2211}}{B}, \quad \ldots\ldots, \quad a_{2212} = \frac{A_{2212}}{2B}$$

and in general

$$(4) \qquad a_{iklm} = \frac{A_{iklm}}{u_{iklm}B}$$

where
$u_{ik,lm} = 1$ if $i = k$ & $l = m$
$u_{ik,lm} = 2$ if $i \neq k$ or $l \neq m$
$u_{iklm} = 4$ if $i \neq k$ & $l \neq m$

But the symbolic coefficients are such that $a_{ik,lm} = a_{lm,ik} = a_{ml,ik}$. Hence must

$$(5) \qquad \frac{A_{ik,lm}}{u_{ik,lm}} = \frac{A_{lm,ik}}{u_{lm,ik}} = \frac{A_{ml,ik}}{u_{ml,lk}}$$

B is a symmetrical determinant & we have in general $A_{ik,lm} = A_{lm,ik} \neq A_{ml,ik}$. Hence relations (5) contrary to hypothesis, impose conditions on the quartic f which proves Theorem I. Conditions (6) are in full

$$(6) \begin{cases} A_{1212} = 4A_{1122} \quad A_{1313} = 4A_{1133} \quad A_{2323} = 4A_{2233} \\ A_{1231} = 2A_{1123} \quad A_{1223} = 2A_{2231} \quad A_{3123} = 2A_{3312} \end{cases}$$

To these, $B \neq 0$ must be added.

Almost immediately follows:

Theorem II. If the quartic f satisfies conditions (6) a quartic F, namely Φ, the evectant of the invariant B, actually exists which has with f, the property ascribed to it by Clifford.

That the quartic F actually exists is evident for we have in (4) determined its coefficients. Moreover these coefficients have been so determined that equation (1) is an identity for both ξ_{ik} & C_{ik} i.e. conditions (6) are sufficient. The contravariant Φ is

$$\begin{vmatrix} \alpha_{1111} & \alpha_{1122} & \cdots & \alpha_{1112} & \xi_1^2 \\ \alpha_{2211} & \alpha_{2222} & \cdots & \alpha_{2212} & \xi_2^2 \\ \cdots & \cdots & \cdots & \cdots & \\ \alpha_{1211} & \alpha_{1222} & \cdots & \alpha_{1212} & \xi_1\xi_2 \\ \xi_1^2 & \xi_2^2 & \cdots & \xi_1\xi_2 & 0 \end{vmatrix}$$

& has been studied by Clebsch* & Scherrer**

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* Ueber Curven vierter Ordnung, Crelle Bd 41

** Ueber ternäre biquadratische Formen, Annali Ser 2, Vol 10

In expanded form, Φ is

$$A_{1.}\xi_1^4 + 2A_{1112}\xi_1^3\xi_2 + 2A_{1113}\xi_1^3\xi_3 + (A_{1212} + 2A_{1122})\xi_1^2\xi_2^2 + 2(A_{1123} + A_{1213})\xi_1^2\xi_2\xi_3 + \cdots$$

Applying conditions (6) we see that the coefficient of $\xi_i\xi_k\xi_l\xi_m$ is exactly the a_{iklm} of f to within the factor B.

An analogous property of the *general* quartic is stated in

Theorem III :- The polar of any point as to the general quartic f has a polar as to Φ which is the original point.

For the proof, see p. 57.

A quartic f satisfying conditions (6) will be called a "special" quartic and we will now derive some of the properties of such quartics. Let first $A'_{ijkl} = \frac{A_{ijkl}}{\mu_{ijkl}}$. Conditions (6) become $A'_{ij,kl} = A'_{ik,jl} = \cdots$ and Φ may be written symbolically, A'^4_ξ.

Theorem IV :- If f is a special quartic, the relation of f & Φ is reciprocal i.e. the Φ of Φ is again f.

For, Φ being A'^{4}_{ξ}

$$\Phi \text{ of } \Phi = \begin{vmatrix} A'_{1111} & A'_{1122} & \cdots & A'_{1112} & x_1^2 \\ A'_{2211} & A'_{2222} & \cdots & A'_{2212} & x_2^2 \\ A'_{1211} & A'_{1222} & \cdots & A'_{1212} & x_1x_2 \\ x_1^2 & x_2^2 & \cdots & x_1x_2 & 0 \end{vmatrix}$$

If the last three columns & last three rows of the determinant be multiplied by 2, the A''s become the A's while $x_i x_\kappa$ for $i \neq \kappa$ obtains a factor 2. Remembering that the minor of A_{ijkl} in the resulting unbordered determinant is $B^4 \alpha_{ijkl}$ we have on developing

$$\Phi \text{ of } \Phi = \frac{B^4}{64} f$$

The reciprocity between f & Φ becomes more marked with the study of their allied forms. Let the conic associate to a line be the conic whose polar is to f is the line repeated & let ψ be the envelope of lines whose associate conics degenerate.

Then the Hessian of Φ is ψ.

For if the conic C be associate to the line l & the second polar of l as to Φ be C_1, f being

a special quartic, C & C_1 coincide. If further C degenerates L is a line both of the Hessian of Φ & of Ψ. This fact may also be seen from the expression of Ψ for the general quartic (Clebsch, loc. cit.)

$$\Psi = \begin{vmatrix} 2\sum_{ik} A_{ik11}\xi_i\xi_k & \sum_{ik} A_{ik12}\xi_i\xi_k & \sum_{ik} A_{ik13}\xi_i\xi_k \\ \sum_{ik} A_{ik12}\xi_i\xi_k & 2\sum_{ik} A_{ik22}\xi_i\xi_k & \sum_{ik} A_{ik23}\xi_i\xi_k \\ \sum_{ik} A_{ik13}\xi_i\xi_k & \sum_{ik} A_{ik23}\xi_i\xi_k & 2\sum_{ik} A_{ik33}\xi_i\xi_k \end{vmatrix}$$

Changing the A's to A''s the coefficients 2 factor out, leaving the expression for the Hessian of $\Phi = A'^{4}_{\xi}$.

Further, the Ψ of Φ is the Hessian of f. For if P is a point of the Hessian of f, the second polar of P is a pair of lines. f being a special quartic the polar of this pair of lines as to Φ is P repeated i.e. the pair of lines is the conic associate to the point P with regard to Ψ. P is then a point of the Ψ of Φ. We have then

Theorem V:- If f is a special quartic

the Hessian of Φ is the Ψ of f:

the Ψ of Φ is the Hessian of f.

Again, let the curve enveloped by the polar conics of Hessian points be called $\mathfrak{D}$. The locus of points into which conics associate to a line break up is the covariant S (Scherrer, loc cit).

Then the $\mathfrak{D}$ of Φ is the S of f

For, let L be a line whose associate conic C breaks up into two points which lie on S. C on f is L^2 & L^2 on Φ is the two points C which by definition lie on the $\mathfrak{D}$ of Φ.

The S of Φ is the $\mathfrak{D}$ of f

For, let C be the degenerate conic (two lines) associate to the point P with regard to Φ. Then P^2 on f is C & C on Φ is P^2. Hence C envelops both the $\mathfrak{D}$ of f & the S of Φ. Summing up we have

Theorem VI. If f is a special quartic
the $\mathfrak{D}$ of Φ is the S of f and
the S of Φ is the $\mathfrak{D}$ of f.

Theorems IV, V, & VI which exemplify the reciprocity between f & Φ may be brought

under a more general theorem. For this let us denote by I an invariantive form (invariant, covariant, contravariant or mixed form) associated to a quantic Q given in point (or line) coordinates, line & point coordinates being supposed interchanged i.e. I of Q is given in line (or point) coordinates. We have then

Theorem VII:- If f is a special quartic
& if the I_1 of Φ is the I_2 of f
the I_2 of Φ is the I_1 of f
to within a factor $\left(\frac{B^4}{6^4}\right)^n$ where n is
the degree of I_1 in the coefficients of Φ.

For, taking Φ in the form A_ξ^4, the I of Φ is a certain function of the A's, x's & ξ's. The I_2 of f is what the I_1 of Φ becomes when the A's are written in terms of the α's. Hence the I_2 of Φ is what the I_1 of Φ becomes when the x & ξ are interchanged & in each A_{mnpq} of I_1, α_{ijkl} is replaced by A_{ijkl}. But this is $\lambda\alpha_{mnpq}$. That λ remains the same

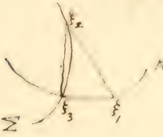
or all coefficients may be seen as follows: $A'_{mn,pq} = \frac{1}{\mu_{mn,pq}} A_{mn,pq}$ where $\mu = 1, 2$ or 4. But if $\mu = 1$, i.e. if $m = n$ & $p = q$ when we replace in $A_{nn,pp}$ the α's by A''s we may suppose instead the α's replaced by A's & factors $\frac{1}{2}$ from each of the last three rows & last three columns are placed outside the minor determinant, i.e. $A'_{nn,pp}$ is replaced by $\frac{1}{64} B^4 \alpha_{nn,pp}$. So if $m = n$ & $p \neq q$ or vice versa, $\mu = 2$ but one less factor $\frac{1}{2}$ can be removed from the minor on replacing the A''s by A's & so on for $m \neq n$ & $p \neq q$. Δ_1 being homogeneous & of degree n in the A''s we have finally that the Δ_2 of Φ is $\left(\frac{B^4}{64}\right)^n \cdot \Delta_1$ of f.

We shall now prove

Theorem VIII: — The necessary & sufficient condition that f be a special quartic is the identical vanishing of a certain covariant conic of degree 5, denoted by C.

We have already found conditions (6) as necessary & sufficient. It will be enough then to show that the condition of the theorem is equivalent to (6). For this it is necessary to calculate the expression for the conic C. There are as is well known two independent C_{320} called by Salmon C_1 & C_2, the first being obtained by operating with σ (C_{204}) on H (C_{360}) the Hessian; the second being obtained by operating with T (C_{306}) on f^2. The sought conic C will be a linear combination of C_1 & C_2.

For simplicity let the quartic f be referred to a triangle of reference chosen as follows:



Take for vertex ξ_3 a point of intersection of the Hessian, H, of f & the Steinerian Σ, of f. Take for ξ_2 the point of Σ which corresponds to ξ_3 considered as a point on H & for ξ_1 the point of H which corresponds to ξ_3 considered as a point on Σ. Then

$\xi_2\xi_3^2 :: f$ & $\xi_3\xi_1^2 :: f$

(:: is a symbol for the expression "is apolar to")

Hence the quartic must take the form

$$f \equiv ax_1^4+bx_2^4+cx_3^4+6hx_1^2x_2^2+12mx_1^2x_2x_3$$
$$+4a_2x_1^3x_2+4b_1x_2^3x_1+4b_3x_2^3x_3+4c_1x_3^3x_1$$

Conditions (6) become

$$ab_3^2c_1^2 = 4ma_2b_3c_1^2-4hm^2c_1^2$$
$$2cmab_1b_3-abcm^2-achb_3-ca_2^2b_3+ch^2m^2=-4m^2a_2b_3c_1+4hm^3c_1$$
(7) $$cma_2^2-achm^2+ba_2^2c_1^2-abhc_1^2+c_1^2h^3+ab_1^2c_1^2-2ha_2b_1c_1^2=-4m^3ac_1$$
$$2h^2b_3c_1^2-2hmb_1c_1^2+2bma_2c_1^2-2a_2b_1b_3c_1^2=acm^2b_3-2cm^3a_2$$
$$abmc_1^2-ab_1b_3c_1^2+h^2mc_1^2-ha_2b_3c_1^2=acm^3$$
$$acm^2b_1-achmb_3+cma_2^2b_3-chm^2a_2=-2am^2b_3c_1$$

Knowledge of the two forms $H=(\alpha\beta\gamma)^2\alpha_x^2\beta_x^2\gamma_x^2$ & $T=(\alpha\beta\xi)^2(\alpha\gamma\xi)^2(\beta\gamma\xi)^2$ is further necessary and it may be worth while to write these expressions for the general quartic which will be written in Salmon's form

$$f = ax_1^4+bx_2^4+cx_3^4+6fx_2^2x_3^2+6gx_3^2x_1^2+6hx_1^2x_2^2+12lx_1^2x_2x_3$$
$$+12mx_2^2x_3x_1+12nx_3^2x_1x_2+4a_2x_1^3x_2+4b_3x_2^3x_3$$
$$+4c_1x_3^3x_1+4a_3x_1^3x_3+4b_1x_2^3x_1+4c_2x_3^3x_2$$

$H = x_1^6(ahg + 2a_2a_3l - a_2^2g - l^2a - a_3^2h) +$

$x_1^5x_2(-2a_2hg + 2b_1ag + 2nah + 2l^2a_2 + 4ma_2a_3 - 2na_2^2$

$-4mla - 2a_3^2b_1) +$

$x_1^5x_3(-2a_3ag + 2mag + 2c_1ah + 2l^2a_3 + 4na_2a_3 - 2a_2^2c_1$

$-4nla - 2a_3^2m) +$

$x_1^4x_2^2(-3h^2g + bag + fah + 4ab_1n - 4ha_2n + 2ga_2b_1 + 2a_2a_3b$

$+2lma_2 - 6la_3b_1 + 3l^2h + 6lma_3 - a_2^2f - 2lb_3a - 4m^2a - a_3^2b) +$

$x_1^4x_3^2(3g^2h + fag + cah - 4gma_3 + 4mc_1a + 2a_3c_1h + 2nla_3 - 6lc_1a_2$

$+2c_2a_2a_3 + 3l^2g + 6lna_2 - a_2^2c - 2alc_2 - 4n^2a - a_3^2f) +$

$x_1^4x_2x_3(-6lhg + 2b_3ag + 2c_2ah - 4a_2hc_1 + 8a_2mg + 4b_1ac_1 - 4b_1a_3g$

$-4man + 8nha_3 - 4nla_3 - 4ma_2l + 4fa_2a_3 + 6l^3$

$-2c_2a_2^2 - 4alf - 2b_3a_3^2) +$

$x_1^3x_2^3(2ab_1f + 2abn - 6nh^2 - 2hgb_1 - 2a_2hf + 2a_2g$ [illegible] $+ 4a_2b_1n$

$-4a_2m^2 - 4b_1l^2 - 4bla_3 - 4mlb_3 + 4ma_3b_3 + 12mnl$

$x_1^3x_2^2x_3(4ab_1c_2 + 6$ [illegible] $g - 6h^2c_1 - 2a$ [illegible] $g - 10a_3hf + 4a_2b$ [illegible] $- 2a_3lg$

$-$ [illegible] $a_2hc_2 +$ [illegible] $a_3lg - 4a_2mn + 4a_3c_1b_1 + 2abc$ [illegible]

$-4a_2fl + 12l^2m - 8b_3a_3l - 4m^2a_3) +$

$x_1^3x_2x_3^2(-6afn + 2bac + 4a_3c_1 + 4a_2c_1m - 8a_2c_2l + 10a_2gf + 6g$ [illegible]

$-6gb_1 - 4ma_3n + 12nl^2 - 12lhc_1 + 8a_3hc_2 - 4b_1ga_3$

$-4la_3f + 4a_3b_1c_1 - 2ca_2h - 4n^2a_2) +$

$x_1^2x_2^2x_3^2(abc - 3af^2 - 3bg^2 - 3ch^2 + 2ab_3c_2 + 2a_3bc_1 + 2a_2bc -$ [illegible]

$[illegible]\, n a_3 b_3 \; 6 b_1 c_1 l + 10 a_2 c_1 b_3 + 10 b_1 a_3 c_2 \; 6 a_2 f n \; 6 b_3 l f$

$-6 c_1 m h - 6 b_1 g n - 6 l h c_2 - 6 m f a_3 + 18 f g h + 18 l m n)$

The remaining terms may be obtained by advancing the subscripts of the variables along with the cyclical advance of the five sets
$a, b, c;\; f, g, h;\; l, m, n;\; a_2, b_3, c_1;\; a_3, b_1, c_2.$

In the same way we may write

$T = \xi_1^6 (bcf - bc_2^2 + 2fb_3c_2 - b_3^2 c - f^3) +$

$\xi_1^5 \xi_2 (6 b_3 m c \; 4 b_3 n c_2 - 2 b_3 c_1 f + 6 f^2 n - 6 f m c_2 + 2 c_2 c_1 b - 4 b_1 f c$
$+ 4 b_1 c_2^2 - 2 b n c) +$

$\xi_1^5 \xi_3 (6 b n c_2 - 4 f b c_1 - 2 f b_1 c_2 + 6 m f^2 - 6 n b_3 f + 2 b_1 b_3 c - 4 m b_3 c_2$
$+ 4 b_3^2 c_1 - 2 b m c) +$

$\xi_1^4 \xi_2^2 (6 f h c + 6 f l c_2 - 3 f^2 g - 6 c_2^2 h + 2 c_2 g b_3 - 6 c l b_3 + c l g \; [illegible]\, n^2 c + 12 c_2 [illegible]$
$+ 6 n c_1 f - 12 n^2 f + 4 n c_1 b_3 - c_1^2 b - 8 b_1 c_1 c_2 + 8 b_1 n c) +$

$\xi_1^4 \xi_3^2 ([illegible]\, c b + 2 h c_2 b_3 - 3 h f^2 - 6 l c_2 b + 6 l g b_3 \; 6 b_3^2 g + 6 l f g + 6 b_1 f n$
$+ 12 m b_3 n + 4 m c_2 b_1 \; b_1^2 c \; 9 b n^2 \; 12 n^2 f \; 8 b_1 c b_3 + 8 [illegible] c_1 [illegible]$

$\xi_1^4 \xi_2 \xi_3 ([illegible]\, f c_2 \; 6 b_3 h c \; [illegible] \; n c_1 [illegible] \; f c_1 + 2 b \; n c \; 12 g f^2 + 8 l b_3 c_2 [illegible]$
$- 6 m n f \; 20 m b_3 c_1 \; [illegible]\, c_2 + 6 b_3 f g + 2 n b c_1 + 12 n^2 b_3) +$

$\xi_1^3 \xi_3^3 (12 b_3 b_1 g - 12 m b g \; 4 c_1 h b \; 4 a_3 b f + 4 a_3 b_3^2 \; 2 c_2 h b_1 + 2 b [illegible] a_2 c_2$
$+ 18 [illegible] - 12 [illegible] \; [illegible]\, b_3 h - [illegible] \; 18 [illegible]$

$+12mhf-2b_3a_2f+8m^3+4b_1^2c_1)+$

$\xi_1^3\xi_2^2\xi_3(-30c_1fh+16c_1lb_3+2c_1bg+24c_1m^2-12c_1b_1n-6gmf-18nb_3g$

$+22b_1c_2g+6a_3f^2-4a_3c_2b_3-2bca_3+30fln+24c_2hn-12n^2m$

$-36mc_1l-10bcl+6mhc-6a_2fc_2+6cb_3a_2)+$

$\xi_1^3\xi_2\xi_3^2(-2a_2cb-4c_2a_2b_3+6a_2f^2-2b_1hc+16lb_1c_2-30b_1fg+24b_1n^2$

$-12b_1c_1m+6a_3c_2b-6a_3fl-36b_3ln+30mlf-10lbc_1$

$-18hc_2m+24mb_3g-12m^2n+6bng-6nhf+22c_1hb_3)+$

$\xi_1^2\xi_2^2\xi_3^2(abc+2b_1a_2c-3h^2c+2a_3c_1b+24b_1c_1l-6mc_1h-20b_3c_1a_2-3g^2b-3$

$-6gnb_1+48fgh-30mg^2-6gb_3l-20a_3c_2b_1+24na_3b_3-6a_3mf$

$+24a_2c_2m-6lhc_2+2ab_3c_2-30n^2h-6na_2f+48lmn-30l^2f)$

Another necessary contravariant is σ given by Salmon (loc. cit., p. 341) for the general quartic

$\sigma \equiv \xi_1^4(bc+3f^2-4b_3c_2)+6\xi_2^2\xi_3^2(ag+gh+2l^2-2a_2n-2a_3m)$

$+12\xi_1^2\xi_2\xi_3(2fl-mn-gb_3-hc_2+b_1c_1)$

$+4\xi_1^3\xi_2(3mc_2-3nf-cb_1+b_3c_1)+4\xi_1^3\xi_3(3nb_3-3mf-bc_1+b_1c_2)+\cdots$

To obtain C we act with σ on H; to obtain C_2 with T on f^2. Employing the above lowered canonical form we find for the coefficient of x_1^2 in C_1 the following

$-a^2b_3^2+6acma_2b-12acm^2-12abhc_1+32am^3c_1$

$+8ab_1^2c^2+5h^3c^2-12ha_2b_1c_1^2+8ha_2^2c_1^2$

and for the coefficient of x_1^2 in C_2 the following

$$-19a^2cd_3 + 2b_1^2c_1^2 - 21abhc_1^2 + 114acma_2b_3 - 21achm^2$$
$$+8am^3c_1 + 72ha_2b_1c_1^2 + 2ba_2^2c_1^2 - 55h^3c_1^2 - 150cm^2a_2^2$$

Let now C be the covariant conic of fifth degree defined by the relation

$$(8) \qquad 150C^* = 19C_1 - C_2$$

The coefficient of x_1^2 in C is

$$ba_2^2c_1^2 - 2cbhm^2 - abhc_1^2 + 4am^3c_1 + ab_1^2c_1^2 + h^3c_1^2 - 2ha_2b_1c_1$$
$$+ cm^2a_2^2$$

But this by reference to the third of conditions (7) is $\qquad A_{2323} - 4A_{2233}$

By a similar calculation we find for the coefficient of x_2x_3 in C $\qquad 2(2A_{1123} - A_{1213})$

We have then finally

$$C = (A_{2323} - 4A_{2233})x_1^2 + (A_{3131} - 4A_{3311})x_2^2 + (A_{1212} - 4A_{1122})x_3^2$$
$$+ 2(2A_{1123} - A_{1213})x_2x_3 + 2(2A_{2231} - A_{1223})x_3x_1 + 2(2A_{3312} - A_{2331})x_1x_2$$

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[*] The only previous mention of C occurs in Salmon (loc. cit., p. 351) where it is merely described and used as a particularly simple combination of the two covariants C_1 & C_2.

evidently conditions (1) are entirely equivalent to $C \equiv 0$ & theorem VIII is proved.

Applying this theorem to equation (1) we have at once

Theorem IX: – If f is a special quartic the two covariant conics of the fifth degree are identical.

This conic C possesses other interesting properties. We have first

Theorem X: If the quartic f has an apolar conic C is the apolar conic expressed in point coordinates.

For if f has an apolar conic $B=0$ and six quantities p_{ik} can be determined such that $A_{ij,kl} = p_{ij}\, p_{kl}$. Moreover (Clebsch, loc. cit.), $\Phi = (p_{11}\xi_1^2 + p_{23}\xi_2\xi_3 + \cdots)^2$ is the square of the apolar conic. Its equation in point coordinates is

$$\begin{vmatrix} p_{11} & \frac{1}{2}p_{12} & \frac{1}{2}p_{13} & x_1 \\ \frac{1}{2}p_{21} & p_{22} & \frac{1}{2}p_{23} & x_2 \\ \frac{1}{2}p_{23} & \frac{1}{2}p_{23} & p_{33} & x_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix} = x_1^2\left(\frac{1}{4}p_{23}p_{23} - p_{22}p_{33}\right) + \cdots$$

$$+ 2x_2x_3\left(\frac{1}{2}p_{11}p_{23} - \frac{1}{4}p_{12}p_{13}\right) + \cdots$$

$= \frac{1}{4}\left[(A_{2323} - 4A_{2233})x_1^2 + \cdots + 2(2A_{1123} - A_{1213})x_2x_3 + \cdots\right]$

i.e. is C as found above

Now the equation in point coordinates of a line conic vanishes identically when & only when the conic is the square of a point. Hence a point taken twice ∴ f when & only when the point is a triple point. We have then

Theorem XI:– The necessary & sufficient conditions that the quartic f have a triple point are $B = 0$ & $C = 0$. φ is then the triple point counted four times.

A quartic with a triple point does not however belong to the above treated class of special quartics for the condition for a special quartic was $B \neq 0$.

Let a conic whose polar as to f has a polar as to φ which is the original conic be called a "fixed" conic [regardless of whether f be a special quartic or not]. Let us recall also that the conic "Associate" to a given conic

y_i is obtained by replacing a row or a column of ξ's in the determinant expression for φ by the coefficient y_i — ξ_1^2 by y_{11}, $\xi_2\xi_3$ by y_{23}."

Having found the conic C we may now derive a formula which will indicate more clearly the number of fixed conics.

Taking $y_{11}x_1^2 + 2y_{23}x_2x_3 + \cdots$ as any conic we have that the polar of y as to Φ is twice

$$\xi_1^2\left[6A_{1111}y_{11} + (A_{1212} + 2A_{1122})y_{22} + (A_{3131} + 2A_{1133})y_{33} + 2(A_{1123} + A_{1231})y_{23}\right.$$
$$\left. + 6A_{1131}y_{31} + 6A_{1112}y_{12}\right] + 2\xi_2\xi_3\left[(A_{1123} + A_{1213})y_{11} + 3A_{2223}y_{22}\right.$$
$$\left. + 3A_{3323}y_{33} + 2(A_{2323} + 2A_{2233})y_{31} + 2(A_{2331} + A_{2312})y_{12}\right] + \cdots$$

The associate conic of y with regard to f is

$$\xi_1^2\left[A_{1111}y_{11} + A_{1122}y_{22} + A_{1133}y_{33} + A_{1123}y_{23} + A_{1131}y_{31} + A_{1112}y_{12}\right] +$$
$$\xi_2\xi_3\left[A_{2311}y_{11} + A_{2322}y_{22} + A_{2333}y_{33} + A_{2323}y_{23} + A_{2331}y_{31} + A_{2312}y_{12}\right] + \cdots$$

Forming now the combination

$\frac{1}{2}$ polar of y − 6 associate of y we obtain

$$\xi_1^2\left[(A_{1212} - 4A_{1122})y_{22} + (A_{3131} - 4A_{1133})y_{33} + 2(A_{1231} - 2A_{1123})y_{23}\right] +$$
$$2\xi_2\xi_3\left[(A_{1213} - 2A_{1123})y_{11} - (A_{2323} - 4A_{2233})y_{23} - (A_{3123} - 2A_{1233})y_{31} - (A_{2312} - 2A_{2231})y_{12}\right] + \cdots$$

The resulting conic is the "intermediate" of y and C. (The intermediate of two conics α_x^2 and β_x^2 is $(\alpha\beta\xi)^2$ the locus of [illegible] the [illegible] conics

in sets of pairs of points). We have then Theorem XII: For the general quartic f and any conic y_x^2, the following relation holds (1) $y \text{ on } \Phi = 2\mathcal{J}_y + 12 A_y$ where $\mathcal{J}_y$ is the intermediate of $(C\, y)$, A_y is the associate of y

If $\mathcal{J}_y \equiv 0$ or if $\mathcal{J}_y \equiv \mu A_y$ then $y \text{ on } \Phi = \lambda A_y$ i.e. y is a fixed conic. But in general $\mathcal{J}_y \not\equiv \mu A_y$, hence must if every conic y is to be a fixed conic, $\mathcal{J}_y = 0$ i.e. $C = 0$, a verification of Theorem VIII.

The general quartic f has, as we shall see later, six fixed conics. Since $C \neq 0$ there must for these conics $\mathcal{J}_y \equiv \mu A_y$. We have then as a first consequence of the above formula (1)

1° The six fixed conics of the general quartic f with regard to Φ are the intermediates of their polars as to f with C

We have further from (1) a proposition stated dualistically by Scherrer (loc. cit.):

2° The polar point of a line L as to Φ is the

polar point of L as to its associate conic

For take γ in (9) to be L^2. $\mathcal{J}_\gamma$ is the two points in which L meets C & L :: these two points. Acting then on (4) with L we have the above statement proved

3° If f has a cusp $\underline{K}$, the cuspidal tangent L is a double line (possibly an inflexional tangent) of Φ.

For $\underline{K}^2$ on f is L^2. Hence $H_{L^2} = \underline{K}^2$. Since $\underline{K}$ lies on L, L is a line of Φ.* $\mathcal{J}_{L^2}$ is the two points in which L cuts C. Then by (1) L^2 on Φ $12\underline{K}^2 + \mathcal{J}_{L^2}$. But since $\underline{K}^2$ & the two points $\mathcal{J}_L$ lie on L the sum $12\underline{K}^2 + \mathcal{J}_{L^2}$ is a binary quartic & breaks up into two points on L the points of contact with Φ of its double line L.

4° If Φ has a double line L, L touches also T, Φ, & C_{12} (12th class)**

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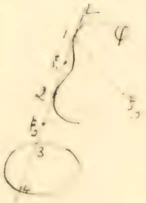
* [illegible] (loc cit) shows that Φ is the locus of lines which lie on their own associate conics

** He further considers C_2.

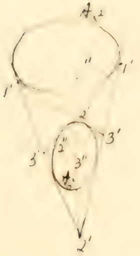


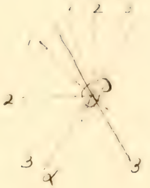
For L^2 on $\phi = 12$ & $\vartheta_{12} = 34$
Hence by (1) $A_{L^2} = \xi_1\xi_2$ i.e. L is a line of ϕ for its associate conic breaks up. And L is a line of C_{12} for it is the join of the points $\xi_1\xi_2$ of a degenerate associate. Taking the triangle of reference as shown & making use of the fact that $\xi_1\xi_2$ on f is L^2 or x_3^2 we have that for $x_3=0$, f becomes $x_1^4+x_2^4$ i.e. L cuts f harmonically & is a line of T.

The condition that C break up is the vanishing of Salmon's invariant E,

5° If the invariant E, of the quartic f vanishes, there is a pencil of fixed conics.

For if C is the two lines $\alpha\beta$ meeting at a point P any two lines through P harmonic with $\alpha\beta$ form a conic whose intermediate as to $\alpha\beta$ vanishes. These conics form a pencil which includes α^2 & β^2. The pencil is then $A_{\alpha^2} + \lambda\beta^2$. The associate pencil is $A_{\alpha^2} + \lambda A_{\beta^2}$. Three conics of this associate





pencil degenerate ; 11', 2'2', 3'3' whose polars are say 11, 22, 33 respectively. Evidently 11'P, 2'2'P, 3'3'P are apolar triads of f & 111'', 222'' & 333'' are apolar triads of C.

6° If C is the square of a line a web of fixed conics exists.

For if C is L^2 the ϑ_γ for any conic γ is the two points in which γ meets L. If in particular γ = LM where M is any line of the plane, $\vartheta_\gamma = 0$ & LM is a fixed conic. The web $A_{L\cdot M}$ (projective to the net) of associate conics contains a single infinity of pairs of points a b such that the polar of a b as to f is L for a point. These points a b are then "conjugate points on S_L"* a curve of third order & degree defined by Caporali* originally

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* Caporali: Memorie di Geometria, p. 343, 12, 4° & p. 344.

is the Jacobian of the net of polar conics of couples of points of L.

Besides the conics of the net a finite number of other fixed conics exist viz: the "pairs of conjugate points as to f" which lie on L. For by definition (Caporali, loc. cit.) a pair of conjugate points as to f is a pair which lie on their polar conic y. If the pair also lie on L, y meets L in this pair i.e. Δ_y & A_y are each this pair of points & y on $\Phi = \lambda A_y$.

Section II

Some Simultaneous Irrational Invariantive Forms of the System: f^n, F^2

The discussion in the preceding section of the number of fixed conics associated to f & its contravariant F suggests naturally this more general problem:

Given a quartic $f = \alpha_x^4$ & a quartic $F = a_\xi^4$ do there exist & if so, in what number, conics whose polar as to f has a polar as to F which is the original conic?

Such conics will also be called "fixed" conics. Suppose then C_ξ^2 is a fixed conic. By definition it must satisfy the following identical equation

$$(1) \qquad C_\alpha^2 \alpha_a^2 a_\xi^2 = \lambda C_\xi^2$$

Equating coefficients of ξ we have the following six equations

$$c_{11}(a_{11}\alpha_{11}a_\alpha^2-\lambda)+c_{22}a_{11}\alpha_{22}a_\alpha^2+\cdots+2c_{23}a_{11}\alpha_{23}a_\alpha^2=0$$
$$c_{11}a_{22}\alpha_{11}a_\alpha^2+c_{22}(a_{22}\alpha_{22}a_\alpha^2-\lambda)+\cdots+2c_{23}a_{22}\alpha_{23}a_\alpha^2=0$$
$$(2) \qquad - - - - - - - - - - - -$$
$$c_{11}a_{23}\alpha_{11}a_\alpha^2+c_{22}a_{23}\alpha_{22}a_\alpha^2+\cdots+c_{23}(2a_{23}\alpha_{23}a_\alpha^2-\lambda)=0$$

Hence in order that C_ξ^2 may exist & satisfy (1) the determinant of the system (2) must

vanish, i.e.

$$\Delta(\lambda) \equiv \begin{vmatrix} a_{11}\alpha_{11}a_\alpha^2-\lambda & a_{11}\alpha_{22}a_\alpha^2 & \cdots\cdots & 2a_{11}\alpha_{31}a_\alpha^2 & 2a_{11}\alpha_{12}a_\alpha^2 \\ a_{22}\alpha_{11}a_\alpha^2 & a_{22}\alpha_{22}a_\alpha^2-\lambda & \cdots\cdots & 2a_{22}\alpha_{31}a_\alpha^2 & 2a_{22}\alpha_{12}a_\alpha^2 \\ - - - & - - - & - - - & - - - & - - - \\ a_{31}\alpha_{11}a_\alpha^2 & a_{31}\alpha_{22}a_\alpha^2 & & 2a_{31}\alpha_{31}a_\alpha^2-\lambda & 2a_{31}\alpha_{12}a_\alpha^2 \\ a_{12}\alpha_{11}a_\alpha^2 & a_{12}\alpha_{22}a_\alpha^2 & \cdots & 2a_{12}\alpha_{31}a_\alpha^2 & 2a_{12}\alpha_{12}a_\alpha^2-\lambda \end{vmatrix} = 0$$

Since f & F are given as entirely general $\Delta(\lambda)=0$ must be viewed as a sextic in the unknown λ with roots $\lambda_1, \lambda_2, \cdots \lambda_6$. Any root λ_i put for λ in the system (2) renders the system consistent & from any five of the equations the ratios of the coefficients $c_{11}, \cdots c_{12}$ may be determined so that the identity (1) for $\lambda=\lambda_i$ is satisfied.

Theorem I:— Corresponding to each root of the equation $\Delta(\lambda)=0$ there exist one fixed conic for the general quartics f & F

But we might as well have asked for conics y_x^2 whose polar as to F has a polar as to f which is equal λy_x^2. An identity would again be obtained which differs from (1) in that the c_i are replaced by the

if that u & x are interchanged. For an interchange of u & x in $\Delta(\lambda)$ merely interchanges rows & columns leaving the equation $\Delta(\lambda)=0$ unaltered. To a root λ_i will as before correspond a conic ${y_x}^2$ which will also be called a fixed conic. Let the point conics just determined be called ${y^1_x}^2, {y^2_x}^2, \dots, {y^i_x}^2$, the line conics previously determined be denoted by ${C^1_\xi}^2, {C^2_\xi}^2, \dots, {C^i_\xi}^2$; so that ${y^i_x}^2$ & ${C^i_\xi}^2$ each correspond to the root λ_i. If the polar of ${C^i_\xi}^2$ as to f be called ${\bar{y}^i_x}^2$, then ${\bar{y}^i_x}^2$ has a polar as to F, $\lambda_i {C^i_\xi}^2$ which has a polar as to f, $\lambda_i {\bar{y}^i_x}^2$ i.e. ${\bar{y}^i_x}^2 \equiv {y^i_x}^2$ or ${y^i_x}^2$ is the polar of ${C^i_\xi}^2$ as to f. Similarly ${C^i_\xi}^2$ is the polar of ${y^i_x}^2$ as to F.

Using the extended definition of fixed conic we have

Theorem II:- Corresponding to each root of the equation $\Delta(\lambda)=0$ there exist two fixed conics ${C^i_\xi}^2$ & ${y^i_x}^2$ which are such that ${C^i_\xi}^2$ & ${y^i_x}^2$ stand in the relation of pole & polar to both f & F

${C^i_\xi}^2$ & ${y^\kappa_x}^2$ are defined by the following identities

$$C^{i\,2}_{\alpha}\, a^{2}_{\alpha}\, a^{2}_{\xi} = \lambda_i C^{i\,2}_{\xi} \qquad y^{\kappa\,2}_{a}\, a^{2}_{\alpha}\, \alpha^{2}_{x} = \lambda_\kappa y^{\kappa\,2}_{x}$$

Operating on the first identity with $y^{\kappa\,2}_{x}$, and the second with $C^{i\,2}_{\xi}$ we have these identities

$$C^{i\,2}_{\alpha}\, a^{2}_{\alpha}\, y^{\kappa\,2}_{a} = \lambda_i C^{i\,2}_{y^\kappa} \qquad y^{\kappa\,2}_{a}\, a^{2}_{\alpha}\, C^{i\,2}_{\alpha} = \lambda_\kappa C^{i\,2}_{y^\kappa}$$

By subtraction we have at once

$$0 = (\lambda_i \; \lambda_\kappa) C^{i\,2}_{y^\kappa}$$

But since in the general case the roots of $\Delta(\lambda)$ are distinct we have $C^{i\,2}_{y^\kappa} = 0$ for $i \neq \kappa$.

Theorem III[*]: Among the fixed conics of the general quartics of F exist the following apolarity relations: $C^{i\,2}_{\xi} : y^{\kappa\,2}_{x}$ for $i \neq \kappa$

Suppose now that a linear identity

(3) $\rho_1 C^{1\,2}_{\xi} + \rho_2 C^{2\,2}_{\xi} + \cdots + \rho_6 C^{6\,2}_{\xi} = 0$ exists. If we multiply the identity (1) for $\lambda = \lambda_i$ by ρ_i & sum for $i = 1, 2, \ldots 6$ we have by virtue of (3) the identity

$$\lambda_1 \rho_1 C^{1\,2}_{\xi} + \lambda_2 \rho_2 C^{2\,2}_{\xi} + \cdots + \lambda_6 \rho_6 C^{6\,2}_{\xi} = 0$$

If again we multiply by (1) for $\lambda = \lambda_i$ by $\lambda_i \rho_i$ & sum for i we have

$$\lambda_1^2 \rho_1 C^{1\,2}_{\xi} + \lambda_2^2 \rho_2 C^{2\,2}_{\xi} + \cdots + \lambda_6^2 \rho_6 C^{6\,2}_{\xi} = 0$$

In the same way we have finally,

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[*] This theorem was proved by Prof. Morley

$$\lambda_1^5\,{}_1C_5^2 + \lambda_2^5\,{}_2C_5^2 + \cdots + \lambda_6^5\,{}_6C_5^2 = 0$$

This succession of identities can subsist only if the determinant

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_6 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_6^2 \\ - & - & - & - & - \\ \lambda_1^5 & \lambda_2^5 & \lambda_3^5 & \cdots & \lambda_6^5 \end{vmatrix} = \prod_{i \neq k} (\lambda_i - \lambda_k) = 0$$

Hence the linear identity (5) can exist only if the roots λ_i & λ_k coincide. Hence

Theorem IV: Among the fixed conics of the general quartics f & F can exist no linear identities.

Having obtained the solution of the problem of the fixed conics in the general case & extracted the relations among them in the above theorems we may view the above configuration in another light. Hereafter then the configuration of conics C_i will be referred to as the "common self polar system of f & F", the definition & consideration of self polar systems in general being reserved to the next section. We state first

Theorem V: The coefficients of the different powers of λ in the equation $\Delta(\lambda)=0$ are simultaneous invariants of f & F i.e. the roots λ_i are simultaneous irrational invariants of f & F.

For consider the identity (1) written $C_\alpha^2 u_\alpha^2 u_\xi^2 - \lambda C_\xi^2 = 0$. Let by virtue of the linear transformation Σ, u_ξ^4 go into $a_\xi'^4$, x_x^4 into $x_x'^4$ & C_ξ^2 into $C_\xi'^2$. Then $C_{\alpha'}'^2 a_{\alpha'}'^2 u_\xi'^2$ (being a symbolic product it is a simultaneous invariant of x_x', $u_\xi'^4$, $C_\xi'^2$) equals $\rho^\kappa C_\alpha^2 a_\alpha^2 a_\xi^2$ & $C_\xi'^2 = C_\xi^2$ by virtue of the substitution Σ of determinant ρ, i.e. we may write

$$C_\alpha^2 u_\alpha^2 u_\xi^2 - \lambda C_\xi^2 = \rho^{-\kappa} C_{\alpha'}'^2 u_{\alpha'}'^2 a_\xi'^2 - \lambda C_\xi'^2$$
$$= \rho^{-\kappa}\left[C_{\alpha'}'^2 u_{\alpha'}'^2 u_\xi'^2 - \lambda' C_\xi'^2\right] \text{ where } \lambda' = \rho^\kappa \lambda$$

Now let λ be a root of $\Delta(\lambda)$ & C_ξ^2 be determined so as to have the originally required property. Then the left hand member is identically zero so also must the right hand member. But on the left C has been determined so as to have an invariantive property with regard to a & α. Hence must its transform C' have on the right the same property with regard to

a' & α'. C' having this property & the right hand member being identically zero it follows that λ' is a root of $\Delta'(\lambda')$ formed for a' & α'

But $\lambda' = \rho^\kappa \lambda \quad \therefore \Delta'(\lambda') = \Delta'(\rho^\kappa \lambda) = C\,\Delta(\lambda)$

In the course of the above proof we see further

Theorem VI: - The conics C_ξ^2 & γ_x^2 of the common self polar system of f & F are simultaneous irrational covariant & contravariants of f & F.

The equation $\Delta(\lambda)=0$ arranged in powers of λ may be written

$$\Delta(\lambda) \equiv \lambda^6 + I_1\lambda^5 + I_2\lambda^4 + I_3\lambda^3 + I_4\lambda^2 + I_5\lambda + I_6 = 0$$

where I_i is a simultaneous invariant of f & F of degree i in the coefficients of each

I_1 is at once seen to be $-(a_{11}\alpha_{11}a_\alpha^2 + \cdots + 2a_{12}\alpha_{12}a_\alpha^2)$ $= a_\alpha^4$ i.e. $I_1 = 0$ is the condition of apolarity of f & F.

I_2 is the sum of fifteen minors of the type

$$\begin{vmatrix} a_{11}\alpha_{11}a_\alpha^2 & a_{11}\alpha_{22}a_\alpha^2 \\ a_{22}\alpha_{11}a_\alpha^2 & a_{22}\alpha_{22}a_\alpha^2 \end{vmatrix} = \begin{vmatrix} a_{11}\alpha_{11}a_\alpha^2 & a_{11}\beta_{22}a_\beta^2 \\ b_{22}\alpha_{11}b_\alpha^2 & b_{22}\beta_{22}b_\beta^2 \end{vmatrix} =$$

$$a_{11}b_{22}\alpha_{11}\beta_{22}(a_\alpha^2 b_\beta^2 - a_\beta^2 b_\alpha^2)$$

or for all permutations of the equivalent symbols

$= \frac{1}{4}(a_\alpha^2 b_\beta^2 - a_\beta^2 b_\alpha^2)($ [illegible] $\alpha_{11}\beta_{22}$ [illegible] α_{22} [illegible] $\alpha_{11}\beta_{22} +$ [illegible] α_{22} [illegible]

summing for all fifteen determinants

$I_2 = \frac{1}{4}(a_\alpha^2 b_\beta^2 - a_\beta^2 b_\alpha^2)^2 = \frac{1}{4}(a_\alpha b_\beta - a_\beta b_\alpha)^2(a_\alpha b_\beta + a_\beta b_\alpha)^2$ (a)

Using now unprimed letters to denote forms of f, primed to denote forms of F we recall that f possesses a mixed form*

$m_2 \equiv \alpha_x^2 \beta_x^2 (\alpha\beta u)^2$ F has then

$m_2' \equiv a_\xi^2 b_\xi^2 (aby)^2$

m_2 on $m_2' = 8[\alpha\beta(ab)]^2[a_\alpha^2 b_\beta^2 + 2a_\alpha b_\alpha a_\beta b_\beta + b_\alpha^2 a_\beta^2]$

Since $[\alpha\beta(ab)] = (a_\alpha b_\beta - a_\beta b_\alpha)$

referring to (a) above we see that

$I_2 = \frac{1}{32}(m_2, m_2')$ where hereafter (X, Y) will denote the result of the operation of X on Y

By a calculation we further obtain

$I_5 = \frac{16}{3}[2(\varphi,\varphi') + (C,C')]$

Recalling the definition of the quantities λ we see that if a value of λ is zero either f or F have an apolar conic; that if either f or F possesses an apolar conic the conic with

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* [illegible], Giornale di Mat. Vol 17 p. 201

the conic whose polar it is as to the other will answer our requirements for $\lambda = 0$. Hence $I_6 = BB'$. These results may be incorporated in

Theorem VII If $\Delta(\lambda)$ be arranged in powers of λ, the coefficient of λ^6 is 1; of λ^5 is $-(f,F)$; of λ^4 is $\frac{1}{32}(m_2, m_2')$; of λ is $\frac{16}{3}[2(\varphi,\varphi')+(CC')]$; of λ^0 is BB'

All the invariants I_K are of a certain general type which is particularly evident when they are exhibited in symbolic determinant form. Thus

$$-I_1 = |a_\alpha^2|^2$$

$$I_2 = \frac{1}{(2!)^2}\begin{vmatrix} a_\alpha^2 & b_\alpha^2 \\ a_\beta^2 & b_\beta^2 \end{vmatrix}^2$$

$$-I_3 = \frac{1}{(3!)^2}\begin{vmatrix} a_\alpha^2 & b_\alpha^2 & c_\alpha^2 \\ a_\beta^2 & b_\beta^2 & c_\beta^2 \\ a_\gamma^2 & b_\gamma^2 & c_\gamma^2 \end{vmatrix}^2$$

$$I_4 = \frac{1}{(4!)^2}\begin{vmatrix} a_\alpha^2 & b_\alpha^2 & c_\alpha^2 & d_\alpha^2 \\ a_\beta^2 & b_\beta^2 & c_\beta^2 & d_\beta^2 \\ a_\gamma^2 & b_\gamma^2 & c_\gamma^2 & d_\gamma^2 \\ a_\delta^2 & b_\delta^2 & c_\delta^2 & d_\delta^2 \end{vmatrix}^2$$

etc the law of formation being obvious. We have already seen that I_1 & I_2 were reducible to the above forms. It will be sufficient to

extend the proof to I_3

I_3 is a sum of twenty minors of the form

$$\begin{vmatrix} a_{11}\alpha_{11}a_\alpha^2 & a_{11}\alpha_{22}a_\alpha & a_{11}\alpha_{33}a_\alpha^2 \\ a_{22}\alpha_{11}a_\alpha^2 & b_{22}\alpha_{22}b_\alpha^2 & b_{22}\alpha_{33}b_\alpha^2 \\ b_{33}\alpha_{11}b_\alpha^2 & a_{33}\alpha_{22}b_\alpha^2 & a_{33}\alpha_{33}a_\alpha^2 \end{vmatrix} = \begin{vmatrix} a_{11}\alpha_{11}a_\alpha^2 & a_{11}\beta_{22}a_\beta^2 & a_{11}\gamma_{33}a_\gamma^2 \\ b_{22}\alpha_{11}b_\alpha^2 & b_{22}\beta_{22}b_\beta^2 & b_{22}\gamma_{33}b_\gamma^2 \\ c_{33}\alpha_{11}c_\alpha^2 & c_{33}\beta_{22}c_\beta^2 & c_{33}\gamma_{33}c_\gamma^2 \end{vmatrix} =$$

$$\alpha_{11}\beta_{22}\gamma_{33}a_{11}b_{22}c_{33}\begin{vmatrix} a_\alpha^2 & a_\beta^2 & a_\gamma^2 \\ b_\alpha^2 & b_\beta^2 & b_\gamma^2 \\ c_\alpha^2 & c_\beta^2 & c_\gamma^2 \end{vmatrix}$$

or on making all possible permutations

$$= \frac{1}{36}\begin{vmatrix} a_\alpha^2 & & \\ & b_\beta^2 & \\ & & c_\gamma^2 \end{vmatrix}\begin{vmatrix} a_{11} & a_{22} & a_{33} \\ b_{11} & b_{22} & b_{33} \\ c_{11} & c_{22} & c_{33} \end{vmatrix}\begin{vmatrix} \alpha_{11} & \alpha_{22} & \alpha_{33} \\ \beta_{11} & \beta_{22} & \beta_{33} \\ \gamma_{11} & \gamma_{22} & \gamma_{33} \end{vmatrix}$$

$$= \frac{1}{36}\begin{vmatrix} a_\alpha^2 & a_\beta^2 & a_\gamma^2 \\ b_\alpha^2 & b_\beta^2 & b_\gamma^2 \\ c_\alpha^2 & c_\beta^2 & c_\gamma^2 \end{vmatrix}\begin{vmatrix} a_{11}\alpha_{11}+a_{22}\alpha_{22}+a_{33}\alpha_{33} & a_{11}\beta_{11}+a_{22}\beta_{22}+a_{33}\beta_{33} & a_{11}\gamma_{11}+a_{22}\gamma_{22}+a_{33}\gamma_{33} \\ b_{11}\alpha_{11}+b_{22}\alpha_{22}+\cdots & b_{11}\beta_{11}+\cdots & b_{11}\gamma_{11}+\cdots \\ c_{11}\alpha_{11}+\cdots & c_{11}\beta_{11}+\cdots & c_{11}\gamma_{11}+\cdots \end{vmatrix}$$

$$= \frac{1}{36}\,\mathfrak{D}\,\overline{\mathfrak{D}}$$

$\mathfrak{D}$ is the same for each of the twenty minors forming I_3. $\overline{\mathfrak{D}}$ varies for each minor & will be multiplied by a factor 1, 2 or 8 as reference to $\Delta(1)$ shows. But these twenty $\overline{\mathfrak{D}}$ may be obtained from $\mathfrak{D}$ as follows: Expand the elements of $\mathfrak{D}$ each column being thereby the sum of six elementary columns. If we take the first three elementary columns of each column of $\mathfrak{D}$ we obtain the particular $\overline{\mathfrak{D}}$

written above. Now D itself equals the sum of the 20 determinants D' obtained by taking the 20 possible choices of three elementary columns of the first column of D for a first column of D', corresponding elementary columns of the second & third column of D forming the second & third column of D'.

But the 20 D''s are the 20 $\overline{D}$'s, the factors 2, 4 & 8 perfectly adjusting themselves. The same argument will manifestly apply up to I_6

Let us consider now an invariant I_7 formed according to the above law

$$I_7 = \frac{1}{(7!)^2}\begin{vmatrix} {a^1_{\alpha_1}}^2 & {a^1_{\alpha_2}}^2 & {a^1_{\alpha_3}}^2 & {a^1_{\alpha_4}}^2 & {a^1_{\alpha_5}}^2 & {a^1_{\alpha_6}}^2 & {a^1_{\alpha_7}}^2 \\ {a^2_{\alpha_1}}^2 & {a^2_{\alpha_2}}^2 & & & & & {a^2_{\alpha_7}}^2 \\ - & & & & & & - \\ {a^7_{\alpha_1}}^2 & {a^7_{\alpha_2}}^2 & & & & & {a^7_{\alpha_7}}^2 \end{vmatrix}$$

Then $I_7 = 0$. For if each element be expanded each column will be made up of six elementary columns and in the expansion according to these elementary columns each determinant will vanish for there are sev

columns & only six elementary columns so that each determinant will necessarily contain elementary columns of the same rank i.e. will have two columns equal to a factor. We have then

Theorem XIII :- The series of simultaneous invariants of f & F formed according to the above law $I_1, I_2, \ldots$ closes with I_6

If now with regard to the quartics f & F we ask for points b_ξ whose polar as to f has a polar as to F which is λb_ξ we have again from the identity

$$b_\alpha a_\alpha^3 a_\xi = \lambda b_\xi$$

that λ is subjected to satisfying an equation $\Delta_1(\lambda) = 0$ (the subscript 1 referring to the class of b_ξ) where

$$\Delta_1(\lambda) = \begin{vmatrix} a_1\alpha_1 a_\alpha^3 - \lambda & a_1\alpha_2 a_\alpha^3 & a_1\alpha_3 a_\alpha^3 \\ a_2\alpha_1 a_\alpha^3 & a_2\alpha_2 a_\alpha^3 - \lambda & a_2\alpha_3 a_\alpha^3 \\ a_3\alpha_1 a_\alpha^3 & a_3\alpha_2 a_\alpha^3 & a_3\alpha_3 a_\alpha^3 - \lambda \end{vmatrix}$$

To each value of λ say λ_1 such that $\Delta_1(\lambda_1) = 0$

there corresponds a desired point*

The coefficients of λ in $\Delta_1(\lambda)$ are again simultaneous invariants of f & t & if we write $\Delta_1(\lambda) = -\lambda^3 + J_1\lambda^2 - J_2\lambda + J_3$ we have by an argument entirely analogous to that used in the previous case of $\Delta_2(\lambda)$ (i.e. $\Delta(\lambda)$ formed for conics) that

$$J_1 = |a_\alpha^3|\,|a_\alpha|$$

$$J_2 = \frac{1}{(2!)^2}\begin{vmatrix} a_\alpha^3 & a_\beta^3 \\ b_\alpha^3 & b_\beta^3 \end{vmatrix}\begin{vmatrix} a_\alpha & a_\beta \\ b_\alpha & b_\beta \end{vmatrix}$$

$$J_3 = \frac{1}{(3!)^2}\begin{vmatrix} a_\alpha^3 & a_\beta^3 & a_\gamma^3 \\ b_\alpha^3 & b_\beta^3 & b_\gamma^3 \\ c_\alpha^3 & c_\beta^3 & c_\gamma^3 \end{vmatrix}\begin{vmatrix} a_\alpha & a_\beta & a_\gamma \\ b_\alpha & b_\beta & b_\gamma \\ c_\alpha & c_\beta & c_\gamma \end{vmatrix}$$

The law of formation of J_i is evident and it may again be seen that invariants of degree higher than 3 formed according to this law vanish identically.

---

* In fact, to every point b_ξ corresponds by the above involute process a unique point b'_ξ. This process is entirely equivalent to a collineation & the desired points are the three fixed points of the collineation

in each column of the second determinant is made up, on expanding the elements, of only three elementary columns & for the same reason as in the case of $\Delta_2(\lambda)$, the determinant of T_κ, $\kappa > 3$, is identically zero.

Let us consider further two series, $f \equiv x_x^{1\,n} = x_x^{2\,n} = \cdots$ & $F \equiv u_\xi^{1\,n} = u_\xi^{2\,n} = \cdots$ & let us seek a ρ-ic, F^ρ, $(\rho < \frac{n}{2}) \equiv b_\xi^\rho$ such that F^ρ on f^n on $F^n \equiv \lambda F^\rho$ i.e. such that the identity

$$(4) \qquad b_\alpha^\rho a_\alpha^{n-\rho} u_\xi^\rho \equiv \lambda b_\xi^\rho \qquad \text{holds.}$$

Equating coefficients of ξ & dividing the coefficients of F^ρ we again have an equation $\Delta_\rho(\lambda) = 0$, where $\Delta_\rho(\lambda)$ is given in the form of a determinant of $\frac{(\rho+1)(\rho+2)}{2}$ rows & columns whose form is entirely analogous to the Δ's previously treated. Moreover in (4) the polynomial coefficients of b_ξ^ρ on the right cut out those of u_ξ^ρ on the left leaving unaffected those of b_α^ρ. Hence the columns of $\Delta_\rho(\lambda)$ are affected by polynomial coefficients as are

have already seen in the case of $\Delta_2(\lambda)$ for quartics. The argument given in full for the I_3 of $\Delta_2(\lambda)$ may now be followed out in detail for the I_K of $\Delta_\rho(\lambda)$ & we have that if $\Delta_\rho(\lambda) = \lambda^{\frac{(\rho+1)(\rho+2)}{2}} - I_1\lambda^{\frac{(\rho+1)(\rho+2)}{2}-1} + \dots \pm I_{\frac{(\rho+1)(\rho+2)}{2}}$

then

$$I_K = \frac{1}{(K!)^2}\begin{vmatrix} a^{1\,n-\rho}_{\alpha_1} & a^{1\,n-\rho}_{\alpha_2} & \dots & a^{1\,n-\rho}_{\alpha_K} \\ a^{2\,n-\rho}_{\alpha_1} & a^{2\,n-\rho}_{\alpha_2} & \dots & a^{2\,n-\rho}_{\alpha_K} \\ a^{K\,n-\rho}_{\alpha_1} & a^{K\,n-\rho}_{\alpha_2} & \dots & a^{K\,n-\rho}_{\alpha_K}\end{vmatrix}\begin{vmatrix} a^{1\,\rho}_{\alpha_1} & a^{1\,\rho}_{\alpha_2} & \dots & a^{1\,\rho}_{\alpha_K} \\ a^{2\,\rho}_{\alpha_1} & a^{2\,\rho}_{\alpha_2} & \dots & a^{2\,\rho}_{\alpha_K} \\ a^{K\,\rho}_{\alpha_1} & a^{K\,\rho}_{\alpha_2} & \dots & a^{K\,\rho}_{\alpha_K}\end{vmatrix}$$

This expression may be simplified withou sacrificing symmetry as follows:
Take for simplicity I_3 which is

$$I_3 \equiv \begin{vmatrix} a_\alpha^{n-\rho} & a_\beta^{n-\rho} & a_\gamma^{n-\rho} \\ b_\alpha^{n-\rho} & b_\beta^{n-\rho} & b_\gamma^{n-\rho} \\ c_\alpha^{n-\rho} & c_\beta^{n-\rho} & c_\gamma^{n-\rho}\end{vmatrix}\begin{vmatrix} a_\alpha^{\rho} & a_\beta^{\rho} & a_\gamma^{\rho} \\ b_\alpha^{\rho} & b_\beta^{\rho} & b_\gamma^{\rho} \\ c_\alpha^{\rho} & c_\beta^{\rho} & c_\gamma^{\rho}\end{vmatrix}\frac{1}{(3!)^2} \equiv D\,\overline{D}\,\frac{1}{(3!)^2}$$

Consider D expanded. The terms T in the expansion are the various permutations of $a, b, \& c$ (or of α, β, γ) in the diagonal term $a_\alpha^{n-\rho} b_\beta^{n-\rho} c_\gamma^{n-\rho}$. To bring any term T to the form of the diagonal term we may of course permute the equivalent symbols

a b c merely inducing however a change in the rows of D which will or will not change the sign of D according as the permutation is odd or even i.e. according as τ has the sign − or +. Hence every term τD is the same as $a_\alpha^{n-\rho} b_\beta^{n-\rho} c_\gamma^{n-\rho} D$ for it may be changed into it by the permutation of equivalent symbols. As there are 3! terms τD we have

$$I_3 = \frac{1}{3!} a_\alpha^{n-\rho} b_\beta^{n-\rho} c_\gamma^{n-\rho} \begin{vmatrix} a_\alpha^\rho & a_\beta^\rho & a_\gamma^\rho \\ b_\alpha^\rho & b_\beta^\rho & b_\gamma^\rho \\ c_\alpha^\rho & c_\beta^\rho & c_\gamma^\rho \end{vmatrix}$$

Similarly D might have been expanded thus interchanging ρ & n−ρ in the final form. The argument may evidently be made general so that we have finally

$$I_\kappa = \frac{1}{\kappa!} a'^{\,n-\rho}_{\alpha_1} \cdots a^{\kappa\, n-\rho}_{\alpha_\kappa} \begin{vmatrix} a'^{\,\rho}_{\alpha_1} \cdots & a^{\kappa\,\rho}_{\alpha_\kappa} \\ \\ a^{\kappa\,\rho}_{\alpha_1} \cdots\cdots & a^{\kappa\,\rho}_{\alpha_\kappa} \end{vmatrix} = \frac{1}{\kappa!} a'^{\,\rho}_{\alpha_1} \cdots a^{\kappa\,\rho}_{\alpha_\kappa} \begin{vmatrix} a'^{\,n-\rho}_{\alpha_1} \cdots\cdots & a'^{\,n-\rho}_{\alpha_\kappa} \\ - & \\ a^{\kappa\, n-\rho}_{\alpha_1} & a^{\kappa\, n-\rho}_{\alpha_\kappa} \end{vmatrix} \qquad (5)$$

All the above results are directly extensible to forms with more variables and we may state the following general

Theorem (A):– Given two q-ary n-ics in complementary variables which symbolically are $f^n = \alpha_{1x}^n = \alpha_{2x}^n = \cdots$ and $F^n = a_{1\xi}^n = a_{2\xi}^n = \cdots$ there exist $N(\rho)$ q-ary ρ-ics $(\rho < \frac{n}{2})$, $F^\rho = b_\xi^\rho$ such that the polar of F^ρ as to f^n has a polar as to F^n which is identically $\lambda_i F^\rho$. These $N(\rho)$ quantities λ_i satisfy an equation in λ of degree $N(\rho)$ whose coefficients are simultaneous invariants of the same degree in the coefficients of f^n & F^n. If the coefficient of λ to the highest power be 1; of the next $-I_1$; of the next, I_2 etc. then the general form of I_κ is given by (5) p. 42 Moreover the series of invariants formed according to the law of the series I_κ closes with $I_{N(\rho)}$ where $N(\rho)$ is the number of terms in the general q-ary ρ-ic.

Another interesting property of the series of invariants I_λ treated in the above theorem will now be derived. For simplicity we agai[illegible] treat the equation $\Delta_2(\lambda) = 0$ (p. 28). The constant term in this equation i.e. the invariant I_6 is, as has already been noticed, the condition that a polar conic (in this case then any conic) of the one quartic should be an apolar conic of the other & breaks up into factors

$I_6 = 64 \mathcal{B}\mathcal{B}'$ Theorem VII.

The property mentioned is described in Theorem IX:— The various invariants in the equation $\Delta_2(\lambda)$ are the results of the successive performance of the operation

$$\mathcal{O} \equiv \frac{2}{5}\Big[\frac{\partial^2}{\partial a_{111}\,\partial \alpha_{1111}} + \frac{1}{4}\,\frac{\partial^2}{\partial a_{1112}\,\partial \alpha_{1112}} + \frac{1}{4}\,\frac{\partial^2}{\partial a_{113}\,\partial \alpha_{113}} + \frac{1}{6}\,\frac{\partial^2}{\partial a_{1122}\,\partial \alpha_{1122}} + \frac{1}{12}\,\frac{\partial^2}{\partial a_{1123}\,\partial \alpha_{1123}} + \cdots\Big]$$

upon the final invariant I_6.

For take I_6 in determinant form i.e. $I_6 = \Delta_2(0)$. In the first place $\mathcal{O}$ will affect only the terms in the principal diagonal of $\Delta_2(0)$. To operate with $\mathcal{O}$ on $\Delta_2(0)$ is to operate success-

ively on each of the six columns of $\Delta_2(0)$ & to add the six resulting determinants. Operating on the elements of the first column reduces all to zero ~~with~~ except the one in the principal diagonal which takes the numerical value 1 [by virtue of the factor $\frac{2}{3}$ affecting Θ] So for the other columns. The six resulting determinants are then the minors of the elements of the diagonal row But these six minors actually make up T_5. So to go one step furth we may operate on $\Delta_2(0)$ with Θ twice by operating with Θ on any two columns of $\Delta_2(0)$, the result of operating on any Colum twice being of course zero through the column. The sum of the 15 resulting determinants is again the sum of the minors of every two elements of the diagonal row i.e. is T_4. The continuation of the proof is obvious.

Taking up now the equation $\Delta(\lambda) = 0$ of p. 38 whose final invariant $J_3 = 0$ is the condition that a polar cubic of the one

quartic is an apolar cubic of the other we have again the various invariants as the results of the successive performance of the operator Θ (with factor $\frac{2}{3}$ changed to $\frac{1}{5}$) upon J_3.

The only objection to the immediate extension of theorem IX to the $\Delta_\rho(\lambda)$ equation formed for ρ-ics with regard to two n-ics, f and F is that the effect of the operator Θ upon the various elements of the diagonal of $\Delta_\rho(0)$ is not obviously the same numerical constant for each element. That this objection does not hold may be shown as follows:

In the first place the general diagonal term is $\frac{\rho!}{\kappa!\,\lambda!\,\mu!}\, a_1^{\kappa} a_2^{\lambda} a_3^{\mu}\, \alpha_1^{\kappa} \alpha_2^{\lambda} \alpha_3^{\mu}\, a_x^{n-\rho}$ $\quad \kappa+\lambda+\mu=\rho$

Expanded this becomes

$$(a)\quad \sum \frac{\rho!}{\kappa!\,\lambda!\,\mu!}\,\frac{(n-\rho)!}{k!\,l!\,m!}\, a_1^{\kappa+k} a_2^{\lambda+l} a_3^{\mu+m}\, \alpha_1^{\kappa+k} \alpha_2^{\lambda+l} \alpha_3^{\mu+m}$$

the summation extending to all values of k, l, m such that $k+l+m=n-\rho$

Now the operator Θ has its various terms affected by the inverses of the polynomial

coefficient of the form of order n. The effect of operating upon the general diagonal term (a) with C is then

$$(b)\quad \sum_{k+l+m=n-\rho} \frac{\rho!}{\kappa!\lambda!\mu!}\,\frac{(n-\rho)!}{k!\,l!\,m!}\,\frac{(\kappa+k)!(\lambda+l)!(\mu+m)!}{n!}$$

We will now show that (b) is independent of the choice of κ, λ, μ provided that $\kappa+\lambda+\mu=\rho$ i.e. that C operating upon a diagonal term gives rise to a numerical constant which is independent of the particular diagonal term chosen. (b) may first be put in the form $\frac{\rho!(n-\rho)!}{n!}\sum_{k+l+m=n-\rho}\frac{(\kappa+k)!}{k!\,\kappa!}\,\frac{(\lambda+l)!}{\lambda!\,l!}\,\frac{(\mu+m)!}{\mu!\,m!}$

The coefficient of x^k in $(1-x)^{-\kappa-1} = \frac{(\kappa+k)!}{\kappa!\,k!}$

" " y^l in $(1-y)^{-\lambda-1} = \frac{(\lambda+l)!}{\lambda!\,l!}$

" " z^m in $(1-z)^{-\mu-1}$ $\frac{(m+\mu)!}{\mu!\,m!}$

Hence the coefficient of $x^k y^l z^m$ in the expansion of $(1-x)^{-\kappa-1}(1-y)^{-\lambda-1}(1-z)^{-\mu-1}$ is $\frac{(\kappa+k)!\,(\lambda+l)!\,(\mu+m)!}{k!\,\kappa!\,\lambda!\,l!\,\mu!\,m!}$

The above summation (b) is the sum of the coefficients of terms homogeneous of degree $k+l+m=n-\rho$ in the same expansion or for $z=y=x$ the above summation is the

coefficient of $x^{n\rho}$ in the expansion of the form $(1-x)^{-(\kappa+\lambda+\mu)-3}$ a coefficient evidently independent of the choice of $\kappa+\lambda+\mu$ if always $\kappa+\lambda+\mu = n$. This proof is manifestly independent of the number of letters k, l, m i.e. theorem IX admits of extension not only to forms of higher degree but to forms with more variables the num

Theorem (B):— The invariant I_{N_ρ} of theorem (A) is the condition when equated to zero that a polar $(n-\rho)$ic of f exist which is :: t θ & viceversa. If $\rho = \frac{n}{2}$, I_{N_ρ} breaks up into two factors, each the condition that one of the forms f θ t have an apolar ρ-ic. The other invariants I_κ of the series of theorem (A) are the results of the successive performance of the operation $\Omega =$

$$c\sum \frac{k!\,l!\,m!\cdots\cdots}{(k+l+m+\cdots)!}\frac{\partial^{2n}}{\partial a_1^k a_2^l a_3^m\cdots\partial \alpha_1^k \alpha_2^l \alpha_3^m\cdots}$$

upon I_{N_ρ} where $(k+l+m+\cdots) = n$ & where c is a constant so adjusted as to make the result of Ω upon a diagonal element of $\Delta_\rho(\lambda)$ unity.

Taking up now the conics of the common self polar system of f^4 & F^4 from the fact of their linear independence proved above we have

Theorem X: Any covariant or contravariant conic of the forms f & F is a linear multiple of the conics of the common self polar system.

In the proof that $C^{i\,2}_{\xi} :: \gamma^{k\,2}_{x}$ for $i \neq k$ we could conclude nothing as to the case $i = k$. We see now that $C^{i\,2}_{\xi} \not{::} \gamma^{i\,2}_{x}$ in general for a conic cannot be apolar to six independent conics.

To actually obtain the conics of the system a root λ_i of the equation $\Delta(\lambda) = 0$ may be put for λ in the system (2) and any five of the equations of the system will give the ratios of the coefficients C_{ik} of the conic $C^{i\,2}_{\xi}$ which correspond to the root λ_i. But the system (2) is thereby treated unsymmetrically. To avoid this difficulty we may, following Hilbert, proceed to consider the modified system (3)

$$
\begin{aligned}
&C^i_{11}(a_{11}\alpha_{11}a_\alpha^2-\lambda_i)+C^i_{22}\,a_{11}\alpha_{22}a_\alpha^2+\cdots+C^i_{12}\cdot 2a_{11}\alpha_{12}a_\alpha^2+x_1^2u=0\\
(3)\quad &C^i_{11}\,a_{22}\alpha_{11}a_\alpha^2+C^i_{22}(a_{22}\alpha_{22}a_\alpha^2-\lambda_i)+\cdots+C^i_{12}\cdot 2a_{22}\alpha_{12}a_\alpha^2+x_2^2u=0\\
&-\ -\ -\qquad\qquad -\\
&C^i_{11}a_{12}\alpha_{11}a_\alpha^2+C^i_{22}\,a_{12}\alpha_{22}a_\alpha^2+\cdots+C^i_{12}(2a_{12}\alpha_{12}a_\alpha^2-\lambda_i)+x_1x_2u=0\\
&C^i_{11}\xi_1^2+C^i_{22}\xi_2^2+\cdots+C^i_{12}\cdot 2\xi_1\xi_2 \qquad -C^{i\,2}_\xi
\end{aligned}
$$

formed from (2) by introducing the variable u hereafter to be determined & by adding the identity last written. The system (3) is made up of seven equations linear & homogeneous in the eight unknowns, C^i_{ik}, u & $C^{i\,2}_\xi$

Denoting by Δ_α the minor of the determinant of the system obtained by striking out the coefficient of the unknown α we have

$$C^i_{11}:C^i_{22}:\cdots:C^i_{12}:u:C^{i\,2}_\xi=\Delta_{C^i_{11}}:\Delta_{C^i_{22}}:\cdots\cdots:\Delta_u:\Delta_{C^{i\,2}_\xi}$$

Hence $u\simeq\Delta_u\simeq\Delta(\lambda_i)=0$ i.e. the system (3) is equivalent to the system (2). Denoting $\Delta_{C^{i\,2}_\xi}$ by $\Delta(x,\xi,\lambda_i)$ we have

$$
C^{i\,2}_\xi\simeq\Delta(x,\xi,\lambda_i)\equiv\begin{vmatrix}
a_{11}\alpha_{11}a_\alpha^2-\lambda_i & a_{11}\alpha_{22}a_\alpha^2 & \cdots\cdots & 2a_{11}\alpha_{12}a_\alpha^2 & x_1^2\\
a_{22}\alpha_{11}a_\alpha^2 & a_{22}\alpha_{22}a_\alpha^2-\lambda_i & \cdots\cdots & 2a_{22}\alpha_{12}a_\alpha^2 & x_2^2\\
-\ -\ - & - - & & & \\
a_{12}\alpha_{11}a_\alpha^2 & a_{12}\alpha_{22}a_\alpha^2 & & 2a_{12}\alpha_{12}a_\alpha^2-\lambda_i & x_1x_2\\
\xi_1^2 & \xi_2^2 & \cdots & 2\xi_1\xi_2 & 0
\end{vmatrix}
$$

If in the same way we seek to determine $y_{i_x}^2$ the a's & x's change places but further the ξ's & x's also so that we have again

$$y_{i_x}^2 \overset{\Omega}{=} \Delta(x, \xi, \lambda_i)$$

$$\therefore \Delta(x, \xi, \lambda_i) \overset{\Omega}{=} y_i C_i$$

Since C on f is y_i & y_i on F is C_i we have

$$\Delta(x, \alpha_{ik}\alpha_x^2, \lambda_i) \overset{\Omega}{=} (y_i)^2$$

$$\Delta(a_{ik}a_\xi^2, \xi, \lambda_i) \overset{\Omega}{=} (C_i)^2$$

These squares will appear in another connection, p. 66.

$\Delta(x, \xi, \lambda)$ is a connex (2,2) which for $\lambda = \lambda_i$ degenerates i.e. breaks up into the product $C_\xi^{i^2} \cdot y_{i_x}^2$

Arranged in powers of λ

$$\Delta(x, \xi, \lambda) \equiv K_0\lambda^5 + K_1\lambda^4 + K_2\lambda^3 + K_3\lambda^2 + \cdots + K_5$$

where K_i is a connex (2,2) which is of degree i in the coefficients of both f & F

Theorem $\underline{XI}$:– If $\Delta(x, \xi, \lambda)$ be arranged in powers of λ, the coefficients of the various powers are invariantive forms of f & F.

The proof of this theorem is entirely analogous to that of Theorem V & will not be repeated.

K_1 is ξ_x^2 the square of the identical connex.
K_1 is $\Delta(x,\xi,0)$ & for a given x is the line conic whose polar as to f has for polar as to F the square of the point x, its meaning for a given ξ being entirely dualistic.

If in the same way the determinant $\Delta_1(\lambda)$ formed with regard to the polars of points & lines as to the quartics f & F be bordered with a column x_1, x_2, x_3 and a row ξ_1, ξ_2, ξ_3 we obtain a determinant $\Delta_1(x,\xi,\lambda)$ whose meaning is entire analogous to the above $\Delta(x,\xi,\lambda)$
For $\lambda = \lambda_\kappa$ where where $\Delta_1(\lambda_\kappa) = 0$
$\Delta_1(x,\xi,\lambda_\kappa) \triangleq b_\xi^\kappa \beta_x^\kappa$ if b_ξ^κ be put for the point whose final polar [i.e. polar as to f on F] is $\lambda_\kappa b_\xi$ β_x^κ for the line whose final polar is $\lambda_\kappa \beta_x^\kappa$
If, again, $\Delta_1(x,\xi,\lambda)$ be arranged in powers of λ the coefficients of these powers are simultaneous invariantive connexes of f & F. The coefficient of λ^2 is the identical connex x_ξ; of λ^0 is a connex which for a given x gives a point whose final

polar as to f & F is x, for a given ξ, a line whose final polar is ξ. The coefficient of λ is the connex $a_\alpha^4 \cdot x_\xi - a_\alpha^3 a_\xi \alpha_x$.

The inquiry for points which coincide with their final polars as to the two ground quartics is coextensive with that for cubics with the same property. For if b_ξ^i on f is $\delta_x^{i\,3}$ & $\delta_x^{i\,3}$ on F is $\lambda_i b_\xi^i$ then is $\delta_x^{i\,3}$ on F, $\lambda_i b_\xi^i$ & $\lambda_i b_\xi^i$ on f is $\lambda_i \delta_x^{i\,3}$. The converse is obvious. There are then three cubics $\delta_x^{i\,3}$ & three others $d_\xi^{i\,3}$ which are reproduced in their final polars. $\delta_x^{i\,3}$ is the polar of b_ξ^i as to f; $d_\xi^{i\,3}$ is the polar of δ_x^i as to F. If we act with $b_\xi^i\ \delta_x^i$ on the product fF we obtain the product $\delta_x^{i\,3} d_\xi^{i\,3}$. Hence

$$\Delta_1(a_x a_\xi^3, \alpha_\ell \alpha_x^3, \lambda_i) \bumpeq \delta_x^{i\,3} d_\xi^{i\,3}$$

The same relations of apolarity exist among the cubics δ_i & d_i as among the points & lines b_i & β_i. The determinant $\Delta_1(a_x a_\xi^3, \alpha_\ell \alpha_x^3, \lambda_i)$ possesses invariant properties exactly similar to those of $\Delta_1(x\, \xi\, \lambda)$. Also it is evident that the

generality of Theorem (A) is not diminished by the assumption $o \leq \frac{1}{2}$.

We have found in the preceding section that under certain conditions f & its contravariant φ possessed an infinite number, a pencil, or even a web of fixed conics. We will now find under what conditions there can be more than six conics which are reproduced in their final polars as to any two quartics f & F.

For every such conic the identity (1) holds & λ must necessarily satisfy the $\Delta(\lambda)$ equation. This equation having for the coefficient of its highest power of λ a numerical constant has always six & only six roots some of which may of course coincide. Having chosen a root & put it in equations (2) the conic is in general uniquely determined by these equations. It may happen that the root chosen λ_1 is such as to make every five rowed minor of $\Delta(\lambda_1)$ vanish. A conic may then be deter

mined from any four of the equations (2) but is no longer determined uniquely as will appear.

Suppose then λ_i makes all the five rowed minors of $\Delta(\lambda)$ zero. Instead of trying to determine a corresponding conic from four of the equations (2) we will form the system (4), a device employed in an analogous problem by Hilbert.

$$(4)\quad \begin{array}{ll} C^i_{11}(a_{11}\alpha_{11}a_\alpha^2-\lambda_i)+\cdots+C^i_{12}\cdot 2a_{11}\alpha_{12}a_\alpha^2+x_1^2u+y_1^2v & =0 \\ C^i_{11}(a_{22}\alpha_{11}a_\alpha^2 \quad +\cdots+C^i_{12}\cdot 2a_{22}\alpha_{12}a_\alpha^2+x_2^2u+y_2^2v & =0 \\ - & \\ C^i_{11}a_{12}\alpha_{11}a_\alpha^2 \quad +\cdots+C^i_{12}(2a_{12}\alpha_{12}a_\alpha^2-\lambda_i)+x_1x_2u+y_1y_2v & 0 \\ C^i_{11}\,\eta_1^2 \quad +\cdots+C^i_{12}\;2\eta_1\eta_2 & =0 \\ C^i_{11}\,\xi_1^2 \quad +\cdots+C^i_{12}\;2\xi_1\xi_2 & -C^{i\,2}_{\xi}=0 \end{array}$$

where u & v are variables to be determined, η is a line of the conic C^i & the last equation is a mere identity. This is a system of 8 equations in the 9 unknowns C^i_{ik}, u, v, & $C^{i\,2}_{\xi}$ each of which is then proportional to a minor of the matrix of the system. But $u = \Delta(\xi\,\eta\,\lambda_i)$ & $v = \Delta(\xi\,\eta\,\lambda_i)$. Since all five rowed minors of $\Delta(\lambda_i)$ vanish we have at once $u=0$, $v=0$ whence the system

(4) is entirely equivalent to the system (2)
Also $C_\xi^{i\,2} \overset{\frown}{=} \Delta(x, y; \xi, \eta; \lambda_i)$ where

$$\Delta(x, y, \xi, \eta, \lambda_i) = \begin{vmatrix} a_{11}\alpha_{11}a_\alpha^2-\lambda_i & \cdots & 2a_{11}\alpha_{12}a_\alpha^2 & y_1^2 & x_1^2 \\ a_{12}\alpha_{11}a_\alpha^2 & \cdots\cdots & 2a_2\alpha_{12}a_\alpha^2-\lambda_i & y_1y_2 & x_1x_2 \\ \eta_1^2 & \cdots\cdots & 2\eta_1\eta_2 & 0 & 0 \\ \xi_1^2 & \cdots\cdots & 2\xi_1\xi_2 & 0 & 0 \end{vmatrix}$$

Moreover $\Delta(x, y, \xi, \eta, \lambda_i) \overset{\frown}{=} C_\xi^{i\,2}$ entirely irrespective of what x & y may be. Hence

$\Delta(x, y, \xi, \eta, \lambda_i) \overset{\frown}{=} C_\xi^{i\,2} \cdot \Phi(y, x)$

However from the dualistic nature of the problem must

$$\Delta(x, y, \xi, \eta, \lambda_i) \overset{\frown}{=} C_\xi^{i\,2} \cdot \Gamma_x^{i\,2}$$

That $C_\xi^{i\,2}$ is not independent of η is seen at once from the system (4) for the next to the last equation requires that η be a line of C^i. We have then that corresponding to a root λ_i of $\Delta(\lambda) = 0$ which makes all the first cond. minors of $\Delta(\lambda)$ vanish there exist a whole pencil of conics C^i for to any given line η corresponds one conic which will touch it. Also to any given point y corresponds a

conic γ_1 which passes through it.

Such a root λ_1 is necessarily a double root of $\Delta(\lambda)=0$ for $\frac{\partial \Delta(\lambda)}{\partial \lambda}$ is a sum of six five rowed minors of $\Delta(\lambda)$ which will all vanish for $\lambda=\lambda_1$. $\Delta(\lambda)$ may of course have a double root without having a pencil of conics corresponding to that root.

In the same way, if a root of $\Delta(\lambda)$, necessarily a triple root makes all the four-rowed minors of $\Delta(\lambda)$ vanish, there will correspond to this root a web of line conics and a net of point conics the product of the web & net being given by the thrice bordered determinant,

$\Delta(x, y, z, \xi, \eta, \zeta, \lambda_2)$.

If finally a sextuple root of $\Delta(\lambda)$ makes all the elements of $\Delta(\lambda)$ vanish, this root will necessarily be the bilinear [illegible] of f & F. Every conic will then coincide with its fixed polar & f & F will stand in the mutual relation of f & Φ discussed in the preceding section on the supposition that

certain six relations were satisfied by f.

These bordered determinants possess the same invariantive character as $\Delta(x, y, \lambda)$ though the forms arising from them, connexes with several sets of variables are more complicated e.g. the coefficient of λ^4 in $\Delta(x, y, \xi, \eta\, \lambda)$ is $x_\xi^2 y_\eta^2 - x_\eta^2 y_\xi^2$

The degenerations of $\Delta_1(\lambda)$ are entirely similar to those of $\Delta(\lambda)$. But contrary to the state of affairs for conics, being given a general quartic f, there exist infinitely many quartics F such that the first polars of every point reproduces the point. Among these quartics are every C_{304} of f. For let $f \equiv \alpha_x^4 = \beta_x^4 = \cdots$, $\sigma \equiv (\alpha\beta\xi)^4 \equiv a_\xi^4 = b_\xi^4 = \cdots$ $\bar{S} \equiv [(\alpha\beta)(\gamma\delta), x]^4 = (ab\xi)^4$ then according to Maisano (loc. cit.) every C_{304} is a member of the pencil $A\sigma + \lambda\varphi$ where $A = (\alpha\beta\gamma)^4$ & $\bar\varphi = (xS\xi)$ The first polar of a point p_ξ as to f & σ is $p_x a_x^3 a_\xi = p_x(\alpha\beta\gamma)^3(\beta\gamma\xi)$ as to f & $\bar\varphi$ is

$p^4_\alpha(\alpha\beta\delta)^3(\beta\delta\xi) = p^4_\alpha(abc)^3(b\delta\xi)$

Now it is shown by Maisano that every symbolic expression which contains the apparent factors $(\alpha\beta\gamma)^3$ or $(abc)^3$ contains the actual factors $(\alpha\beta\gamma)^4$ or $(abc)^4$ respectively. Hence the above final polar become to within numerical factors

$(\alpha\beta\gamma)^4 p^4_\xi$ & $(abc)^4 p^4_\xi$

∴ The final polar of every point as to f & any member of the above pencil is the original point. We have proved then theorem III of the first section since Φ is a member of the above pencil. We have also

Theorem XII: $B=0$ is the condition that every polar cubic of f :: Φ & viceversa

The corresponding theorem for f, ϕ & $A=0$ is stated by Maisano (loc cit)

Section III

Systems of Conics Connected with the Quartic

It has been said a characteristic feature of forms of even order $2n$ is the one to one correspondence set up by them between the aggregate of forms of order n & that of forms of class n. To a certain extent then point & line conics are related to a given quartic $f = a_x^4$ as are points & lines to a given conic. In the following some of these relations will be considered. Line conics will hereafter be taken without polynomial coefficients, point conics with polynomial coefficients.

Theorem I:– If $C^2_{1\,\xi}$ :: polar conic of $C^2_{2\,\xi}$,
$C^2_{2\,\xi}$ :: polar conic of $C^2_{1\,\xi}$

For by hypothesis $C^2_{1\,\xi} \cdot C^2_{2\,\xi} : f$ which is a perfectly mutual relation between C^1 & C^2.

The system of conics $\left.\begin{matrix} y^{1\,2}_x, y^{2\,2}_x, \dots, y^{6\,2}_x \\ C^{1\,2}_\xi, C^{2\,2}_\xi, \dots, C^{6\,2}_\xi \end{matrix}\right\}$ will be called an "apolar system" if $y^{i\,2}_x :: C^{k\,2}_\xi$ for $i \neq k$ & $y^{i\,2}_x \not{:}\,: C^{i\,2}_\xi$. The apolar system is an immediate extension of the three point

& three lines forming a triangle. Such a system may be formed by choosing the six point conics as linearly independent but otherwise arbitrary. The line conics will then be determined. For we have by identity

$$C'_{11}\xi_1^2 + C'_{22}\xi_2^2 + C'_{33}\xi_3^2 + C'_{23}\xi_2\xi_3 + C'_{31}\xi_3\xi_1 + C'_{12}\xi_1\xi_2 = 0.$$

by apolarity, $C'_{11}y^2_{11} + C'_{22}y^2_{22} + C'_{33}y^2_{33} + C'_{23}y^2_{23} + C'_{31}y^2_{31} + C'_{12}y^2_{12} = 0$

— — — — — — — — — — — —

$$C'_{11}y^6_{11} + C'_{22}y^6_{22} + C'_{33}y^6_{33} + C'_{23}y^6_{23} + C'_{31}y^6_{31} + C'_{12}y^6_{12} = 0$$

$$\therefore C'^{2}_{\xi} = \Gamma^1_{11}\xi_1^2 + \Gamma^1_{22}\xi_2^2 + \Gamma^1_{33}\xi_3^2 + \Gamma^1_{23}\xi_2\xi_3 + \Gamma^1_{31}\xi_3\xi_1 + \Gamma^1_{12}\xi_1\xi_2 = 0$$

& so for $C^{\kappa\,2}_{\xi}$ where Γ^{κ}_{mn} is the minor of y^{κ}_{mn} in the determinant Δ

$$\Delta \begin{vmatrix} y'_{11} & \cdots & y'_{12} \\ y^6_{11} & \cdots & y^6_{12} \end{vmatrix}$$

Theorem II. If an apolar system is such that its point conics are the polars as to f of the line conics $C^{\kappa\,2'}_{\xi}$ & its line conics have for polars as to f the point conics $y^{\kappa\,2'}_{x}$, the conics C^{κ} & y^{κ} also form an apolar system.

The proof is immediate from Theorem I.

In a certain sense the one apolar system is the polar of the other. Two apolar systems

may coincide with its polar. If then an apolar system is such that the point conics are the polars of the line conics as to a quadric f, the apolar system will be called a "self-polar system of f". We have already met with such systems in the preceding section.

In order that an apolar system be a self polar system of f, certain relations must exist among the point conics previously arbitrary which may be determined as follows. Suppose the polar of $C^{\kappa\,2}_{\xi} = \Gamma^{\kappa\,2}_{\xi}$ as to f is $\lambda_\kappa y^{\kappa\,2}_x$

i.e. (1) $\Gamma^{\kappa\,2}_{\alpha}\alpha_x^2 \equiv \lambda_\kappa y^{\kappa\,2}_x$ $\kappa = 1, 2, \cdots, 6$

Equating coefficients of x_1^2 for $\kappa = 1, \cdots, 6$ we obtain the system of equations

$$\alpha_{11}\Gamma^1_{11} + \alpha_{1122}\Gamma^1_{22} + \alpha_{1133}\Gamma^1_{33} + \alpha_{1123}\Gamma^1_{23} + \alpha_{1131}\Gamma^1_{31} + \alpha_{1112}\Gamma^1_{12} = \lambda_1 y^1_{11}$$

- - -

$$\alpha_{11}\Gamma^6_{1} + \alpha_{1122}\Gamma^6_{22} + \alpha_{1133}\Gamma^6_{33} + \alpha_{1123}\Gamma^6_{23} + \alpha_{1131}\Gamma^6_{31} + \alpha_{1112}\Gamma^6_{12} = \lambda_6 y^6_{11}$$

From these we may obtain the values of α_{11ik}. Noting that the determinant of the system is that of the first minors of Δ we have that is a factor Δ^4

$$(2)\quad \begin{aligned} \alpha_{1111} &= \lambda_1 {y^1_{11}}^2 + \lambda_2 {y^2_{11}}^2 + \cdots + \lambda_6 {y^6_{11}}^2 \\ \alpha_{1122} &= \lambda_1 y^1_{11} y^1_{22} + \lambda_2 y^2_{11} y^2_{22} + \cdots + \lambda_6 y^6_{11} y^6_{22} \\ &\text{— — — — — — —} \\ \alpha_{1112} &= \lambda_1 y^1_{11} y^1_{12} + \lambda_2 y^2_{11} y^2_{12} + \cdots + \lambda_6 y^6_{11} y^6_{12} \end{aligned}$$

In the same way by the use of the other powers of x in (1) we have in general

$$(3)\qquad \alpha_{ik,lm} = \sum_1^6{}^{\rho} \lambda_\rho y^\rho_{ik} y^\rho_{lm}$$

Hence already $\alpha_{ik,lm} = \alpha_{lm,ik}$. But necessarily $\alpha_{1122} = \alpha_{1212}$ & $\alpha_{1123} = \alpha_{1213}$ with the four other cyclical equations. We must have therefore

$$(4)\quad \begin{aligned} &\lambda_1 (y^1_{11} y^1_{22} - {y^1_{12}}^2) + \cdots + \lambda_6 (y^6_{11} y^6_{22} - {y^6_{12}}^2) = 0 \\ &\lambda_1 (y^1_{11} y^1_{23} - y^1_{12} y^1_{13}) + \cdots\cdots + \lambda_6 (y^6_{11} y^6_{23} - y^6_{12} y^6_{13}) = 0 \end{aligned}$$

with the other equations obtained by cyclical advance of subscripts.

If the point conics have been chosen as repeated lines, equations (4) are identically satisfied. If not, equations (4) mean that a linear identity $\sum_1^6{}^{i} \lambda_i (y_i y_i' \xi)^2 = 0$ exists among the line equations of the point conics.

Theorem III:– If an apolar system is a self polar system of a quartic, a linear iden

tity must necessarily exist among the line equations of the point conics.

These self polar systems may be constructed for the given quartic f as follows:- Take the conic $C_{\xi}^{1\,2}$ arbitrarily & its polar as to f, $\gamma_{x}^{1\,2}$; take $C_{\xi}^{2\,2}$ :: $\gamma_{x}^{1\,2}$, otherwise arbitrary, & its polar $\gamma_{x}^{2\,2}$; take $C_{\xi}^{3\,2}$ :: $\gamma_{x}^{1\,2}$ & $\gamma_{x}^{2\,2}$ & linearly independent of C_1 & C_2, & its polar $\gamma_{x}^{3\,2}$. Continuing in this way up to (5) we have finally C_6 completely determined by the five preceding γ's. Its polar γ_6 is also apolar to the five preceding C's by theorem I. A linear identity exists among the line equations of the point conics. In the choice of the conics C we suppose such conics as are apolar to their own polar, for otherwise the sixth point & line conic would be linearly dependent on the others.

Theorem III shows how the sixth point conic is dependent on the first five. The sixth line conic must also be dependent on the other five. The nature of this dependence may

obtained as follows: From the apolarity conditions we have the fifteen equations

$$\sum_{ik,lm} x_{ik\,lm} C^{\rho}_{ik} C^{\sigma}_{lm} = 0 \quad \rho \neq \sigma \qquad \begin{matrix} ik \,\&\, lm = 11, 22, \ldots, 12 \\ \rho \,\&\, \sigma = 1, 2, \ldots, 6 \end{matrix}$$

To these may be added the six

$$\sum_{ik,lm} \alpha_{ik,lm} C^{\rho}_{ik} C^{\rho}_{lm} = \tau_{\rho}$$

From these 21 non homogeneous equations in the 21 unknowns $\alpha_{ik,lm} = \alpha_{lm,ik}$ we may obtain the values of $\alpha_{ik,lm}$ & expressing that $x_{ik\,lm} = x_{il,mk}$ there results six relations among the coefficients C. These are however of high degree. In fact it could not be expected that the line conics are related in the same way as the point conics for the correspondence between them is set up by a point quartic. But in the case of a self polar system of a line quartic which is also a self polar system of f (the common self-polar system of the preceding section), the point equations of the line conics are also connected by a linear identity

Theorem IV: The quartic f is linearly expressible as a sum of the squares of the conics

conics of any self polar system.

For being a self polar system, the necessary relations (7) hold. From equations (3) the theorem is then obvious.

This includes the expression of the quartic as a sum of the fourth powers of linear forms. For it is only necessary to choose the point conics as the squares of lines. The manner of choice is also indicated by the theorem. The first line is chosen at random. Since the square of the second line is to be ∴ associate conic of the first it must lie upon it; for the same reason the four other lines must be the four common lines of the first two associate conics. Also the first two lines must not be self associate as was previously noted for the general self polar system.

Theorem $\overline{V}$:- A point quartic f is expressible as a sum of squares of six point conics, a line quartic ϕ as a sum of six line conics, the point & line conics together forming

a self polar system of both f & F. This reduction is unique.

For from the preceding section the two quartics possess but one common self polar system.

The following facts may now be noted:

1° Having fixed the homogeneous coefficients in a self polar system of f, in the expression $f = \sum_1^4 \rho_\kappa (y_x^{\kappa 2})^2$ the coefficients ρ_κ are such that to a factor independent of κ, $C_f^{\kappa 2}$ or $f \equiv \rho_\kappa y_x^{\kappa 2}$.

2° The coefficients in the linear identity of theorem III are these same ρ's.

3° The polar reciprocal of C_f^2 as to $y_x^2\, y_x'^2$ is

$$y_c y_c' y_x y_x' \equiv \mathcal{P}_{c,y}.$$

4° The intermediate of C_f^2 & $(y\,y')^2$ is

$$C_y^2 \cdot y_x^2 - \mathcal{P}_{c,y} \equiv \partial_{c,y}$$

5° The polar of C_f^2 as to $(y_x^2)^2$ is

$$y_c^2 \cdot y_x^2 + 2\,\mathcal{P}_{c,y} = 3\, y_c^2 \cdot y_x^2 - 2\,\partial_{c,y}$$

6° $\rho_1 \partial_{c,y_1} + \rho_2 \partial_{c,y_2} + \cdots = \partial_{c,\,\rho_1 y_1 + \rho_2 y_2 +}$

7° $y_{C_\kappa}^{\kappa 2}$ is the Δ of page 11 & is independent of κ

Fixing definitely the coefficients of the conic of the common self polar system of f & F we have

$$(5)\qquad f \equiv \pi_1\left(y_x^{\overset{2}{1}}\right)^2 + \cdots\cdots + \pi_6\left(y_x^{\overset{2}{6}}\right)^2$$

$$F \equiv \rho_1\left(C_\xi^{\overset{2}{1}}\right)^2 + \cdots\cdots + \rho_6\left(C_\xi^{\overset{2}{6}}\right)^2$$

where π_κ & ρ_κ are defined by 1° & 2°

By 4°, 5°, 6° the polar of $C_\xi^{\overset{2}{1}}$ as to f is

$$3\pi_1\, y_{C_1}^{\overset{2}{1}} \cdot y_x^{\overset{2}{1}} - 2\,\partial_{C_1,\ \pi_1 y_1 + \cdots + \pi_6 y_6}$$

But since by 4° $\partial_{c,y}$ is formed with reference to the line equation of y & by 2°, $\sum \pi_\kappa (y\, y'_\xi)^2 = 0$, also by 7° $3y_{C_1}^{\overset{2}{1}}$ may be neglected we have the polar of $C_\xi^{\overset{2}{1}}$ as to f is $\pi_1 y_x^{\overset{2}{1}}$ similarly the polar of $\pi_1 y_x^{\overset{2}{1}}$ as to F is $\pi_1\rho_1 C_\xi^{\overset{2}{1}}$

The connection between the roots of the $\Delta(\lambda)$ equation of f & F & the expression (5) gives

Theorem VI:- If according to theorem V the quartics f & f' are put in the form (1) the roots of the $\Delta(\lambda)$ equation of f & f' are such that to a factor independent of κ,

$$\lambda_\kappa = \pi_\kappa \rho_\kappa$$

Let us finally suppose two general point quartics given, $f \equiv \alpha_x^4$ & $f' \equiv \beta_x^4$ & let us ask for conics C_ξ^2 whose polar as to f & as to f' coincide or whose polar as to $f \equiv -\lambda$ times its polar as to f'. Then must $C_\alpha^2 \alpha_x^2 \equiv -\lambda C_\beta^2 \beta_x^2$ or
This identity gives to determine C_{ik} six equation

$$(2) \quad C_{11}(\alpha_{11ik} + \lambda\beta_{11ik}) + \cdots + C_{23}(\alpha_{23ik} + \lambda\beta_{23ik}) + \cdots = 0$$

We must have then $\Delta'(\lambda) = 0$ where

$$\Delta'(\lambda) \equiv \begin{vmatrix} \alpha_{1111} + \lambda\beta_{1111}, & \cdots & , \alpha_{1112} + \lambda\beta_{1112} \\ - & - & - \\ \alpha_{1211} + \lambda\beta_{1211}, & \cdots\cdots & , \alpha_{1212} + \lambda\beta_{1212} \end{vmatrix}$$

$\Delta'(\lambda) = 0$ gives then six values of λ which substituted in (2) renders them consistent re

the determination of c_{ik}.

There are then six line conics $C_\xi^{1^2}, \cdots, C_\xi^{6^2}$ each of which has the same polar as to f & f′. Thus

$$C_\alpha^{i^2}\alpha_x^2 = -\lambda_i C_\beta^{i^2}\beta_x^2$$

$$C_\alpha^{k^2}\alpha_x^2 = -\lambda_k C_\beta^{k^2}\beta_x^2$$

Operating on the first identity with $C_\xi^{k^2}$, on the second with $C_\xi^{i^2}$ & subtracting we have

$$(\lambda_i - \lambda_k)\, C_\beta^{i^2} C_\beta^{k^2}$$

Since λ_i is not in general equal to λ_k for $i \neq k$ $C_\beta^{i^2} C_\beta^{k^2} = 0$ i.e. $C_\xi^{k^2}$ :: polar conic of $C_\xi^{i^2}$ as to f & viceversa. Hence the six line conics with their polars form a common self polar system of f & f′.

Theorem VII:- Any two point quartics f & f′ are simultaneously expressible in but one way as the sum of squares of the same six point conics, the associate line conic of a point conic being the same for both f & f′ and the point & line conics together forming a self polar system of both f & f′.

We have in theorem II a necessary condition that an apolar system be a self polar system of a quartic. That no others exist may be seen as follows:– An apolar system depends on thirty constants. If it be a self polar system of some (undetermined) quartic, the line equation of one point conic is determined linearly in terms of the other five which cuts out five constants while the ratios of the five constants in the identity among the first five & the last brings in four more constants. A self polar system of an undetermined quartic involves 29 constants.
But being given a quartic f we see from page 64 that the construction of a self polar system of f depends on 13 constants which added to the 14 arbitrary constants which appear in f gives the 29 previously found.

Included in the aggregate of self polar systems of f (13 times infinite) is an aggreg (14 times infinite) of systems which f has

in common with line quartics. They are distinguished further by the existence of a linear identity among the point equations of the line conics. It includes also the aggregate (14 times infinite) of systems which has in common with other point quartics f, which are distinguished by the existence of *two* linear identities among the line equations of the point conics. These last will then be called "restricted" self polar systems.

Returning to the equation $\Delta(\lambda)=0$, the coefficients of the various powers of λ in this equation are well known being the result of the successive performance of Aronhold's differential operation $\sum \beta_{iklm}\frac{\partial}{\partial\alpha_{iklm}}$ on B the catalecticant of f.

In fact $\Delta(\lambda)$ is merely the catalecticant of the pencil of quartics $f+\lambda f'$ and the roots of $\Delta(\lambda)=0$ are the values of λ for which a quartic of the pencil has an absolute conic. Thus we

should expect for if a conic acting on $f+\lambda f'$ has no polar, its polar as to f must be $-\lambda$ times its polar as to f'

If we form the contravariant Φ of the pencil we obtain a line quartic which involves λ to the fifth power & which for $\lambda=\lambda_1, \dots \lambda=\lambda_6$ ($\lambda_1, \dots \lambda_6$ being the roots of $\Delta(\lambda)=0$) breaks up into $(C_{1\xi}^2)^2, \dots (C_{6\xi}^2)^2$. Hence Φ must in general be of the form

$$(\lambda-\lambda_2)\cdots(\lambda-\lambda_6)(C_{1\xi}^2)^2+(\lambda-\lambda_1)(\lambda-\lambda_3)\cdots(\lambda-\lambda_6)C_{2\xi}^2+\cdots+(\lambda-\lambda_1)\cdots(\lambda-\lambda_5)(C_{6\xi}^2$$

Also C, the $C_{,26}$ which is the point equation of an apolar conic must be of the form

$$(\lambda-\lambda_2)\cdots(\lambda-\lambda_6)(C_1C_1'x)^2+\cdots+(\lambda-\lambda_1)\cdots(\lambda-\lambda_5)(C_6C_6'x)^2$$

But for $\lambda=0$, these forms reduce to the corresponding forms of f. Hence the

$$\text{(6)}\quad \Phi \text{ of } f = \lambda_2\lambda_3\lambda_4\lambda_5\lambda_6(C_{1\xi}^2)^2+\cdots+\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5(C_{6\xi}^2)^2$$

$$C \text{ of } f = \lambda_2\lambda_3\lambda_4\lambda_5\lambda_6(C_1C_1'\xi)^2+\cdots+\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5(C_6C_6'\xi)^2$$

But the same form will be obtained by the use of any quartic f' i.e. $C_1, \dots C_6$ may be the conics of any restricted self polar system of f.

Theorem VIII. The contravariant Φ is expressible as a sum of squares of the six line conics of any restricted self polar system of f. The covariant C is expressible as a sum of the reciprocals of those line conics

Since (6) holds for any restricted self polar system we may regard C_ξ^2 as any conic (except such as :: their own polar), say the conic whose polar is y_x^2. Then

$$y \text{ on } \Phi = 4 \frac{\Pi\lambda_i}{\lambda_1} C_{1y}^2 . C_{1\xi}^2 + 8 \sum_{1}^{6} \frac{\Pi\lambda_i}{\lambda_\kappa} C_{\kappa y} C'_{\kappa y} C_\xi C'_\xi \quad \text{by } 5$$

$$\text{and } \vartheta_{C,y} = 2 \quad '' \quad -2 \quad '' \quad \text{by } 4$$

$$\therefore y \text{ on } \Phi + 4\vartheta_y = \lambda A_y$$

a result which agrees to within numerical constants with that found by actual calculation (see p 22.) though in this connection it is not proved for every conic.

Translation to a Space of Five Dimensions

Let the line conics of the plane be represented by points in a five-dimensional space, S_5. As is well known*, a point conic is then represented by a plane 4-spread, S_4', or simply S_4, in this S_5. Degenerate line conics are represented by the points of a cubic 4-spread $\mathfrak{M}_4^3$. On $\mathfrak{M}_4^3$ lies a 2-spread $\mathfrak{F}_2^4$, the map of the doubled points of the plane. Dualistically degenerate point conics map into the planes enveloping a 4-spread of third class M_4^3. In M_4^3 is included a 2-spread of planes F_2^4, the map of the double lines of the plane. The map of a line conic C is a point whose polar plane as to $\mathfrak{M}_4^3$ is the map of a point conic C'. C' is then C expressed in line coordinates.

The results previously obtained algebraically are now translatable as follows:

The correspondence between point & line conic set up by a quartic f translates into a correspondence between point & S_4 in the S_5 which is there set up by a special pan

quadric 4-spread, Q_4^2. Q_4^2 cuts out on F_2^4 a curve in S_5, an f_1^8. An apolar system maps into a "basis" or "tetrahedroid" in S_5. + self polar system of f maps into a "self conjugate or self polar" basis with regard to Q_4^2 i.e. the polar of a vertex of the basis as to Q_4^2 is the S_4 through the other five vertices. Common self polar systems of two quartics map into the common self conjugate bases of the quadrics which are the maps of the quartics. Theorem III is translated as follows:-

Every self conjugate basis of a Q_4^2 which is the map of a quartic curve is such that the six polar points of its planes as to M_4^3 lie in an S_4

The translations of the other theorems lie at hand

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* Segre, Atti Acc. Torino, t. XX, 1885

Aschieri Fondamenti, Mem. Inst. Lombardo, 1873.
Aschieri, Mem. Inst. Lomb., 1879
Study, Annalen, t. 39.

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Vita.

Arthur Byron Coble was born Nov. 3 1878 in Williamstown, Pa. He entered Pennsylvan College Gettysburg in the fall of 1893 being graduated from thence in 1897. After teaching for one year in the public schools of Lykens Pa. he entered the Johns Hopkins University where he remained up to the present. During the year 1900-01 he was a student assistant, during the year 1901-2, Fellow, in Mathematic His course at the University included Mathematics as the principal subject with Physics & Astronomy as first & second subordinate subjects respectively